Noncommutative pressure and the variational principle in Cuntz–Krieger–type C^*–algebras

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Abstract

Let a be a self-adjoint element of an exact C^*–algebra $\mathcal{A}$ and $\theta : \mathcal{A} \to \mathcal{A}$ a contractive completely positive map. We define a notion of dynamical pressure $P_\theta(a)$ which adopts Voiculescu's approximation approach to noncommutative entropy and extends the Voiculescu–Brown topological entropy and Neshveyev and Størmer unital-nuclear pressure. A variational inequality bounding $P_\theta(a)$ below by the free energies $h_\sigma(\theta) + \sigma(a)$ with respect to the Sauvageot–Thouvenot entropy $h_\sigma(\theta)$ is established in two stages via the introduction of a local state approximation entropy, whose associated free energies function as an intermediate term.

Pimsner C^*–algebras furnish a framework for investigating the variational principle, which asserts the equality of $P_\theta(a)$ with the supremum of the free energies over all θ–invariant states. In one direction we extend Brown's result on the constancy of the Voiculescu–Brown entropy upon passing to the crossed product, and in another we show that the pressure of a self-adjoint element over the Markov subshift underlying the canonical map on the Cuntz–Krieger algebra $\mathcal{O}_A$ is equal to its classical pressure. The latter result is extended to a more general setting comprising an expanded class of Cuntz–Krieger-type Pimsner algebras, leading to the variational principle for self-adjoint elements in a diagonal subalgebra. Equilibrium states are constructed from KMS states under certain conditions in the case of Cuntz–Krieger algebras.

1   Introduction

Let (X, θ) be a compact topological dynamical system. The variational principle in classical ergodic theory, established in full generality by Walters [31], asserts that the topological pressure of a real-valued continuous function a on X is the supremum of the free energies, i.e., of sums of the form $h_\sigma(\theta) + \sigma(a)$ where σ ranges over all θ–invariant measures of X and $h_\sigma(\theta)$ stands for the (Kolmogorov–Sinai) measure-theoretic entropy of θ. Topological pressure was introduced by Walters [31] as a dynamical abstraction of the statistical mechanical concept of pressure defined as the logarithmic partition function density under a thermodynamic limit. Adapting the approaches of Adler, Konheim, and McAndrew [1], Bowen [6], and Dinaburg [12] to topological entropy, Walter's definition functions not by invoking a specific sequence of finite subsystems, as in the thermodynamic notion of entropy or pressure density, but rather samples over the dynamical limits of all finite subsystems. Thus, for a lattice system, the thermodynamic limit is reconceptualized as a dynamical limit with space translation generating the sequence of subsytems, and the variational principle for translation-invariant lattice systems (see Ruelle [22]) is subsumed into Walters' general result.

Our ultimate goal is to investigate the variational principle in a noncommutative dynamical setting which, in analogy to the classical case, captures the shift-invariant lattice system model of quantum thermodynamics as a special instance. Compared to the topological situation, noncommutative dynamics presents a much less definitive state of affairs for a theory of entropy and pressure. For instance, various alternative notions of entropy are available, from Voiculescu's approximation definition [29] to Connes, Narnhofer, and Thirring's [9] and Sauvageot and Thouvenot's [26] physically motivated approaches in which the system is observed via Abelian models (see below). Recently Størmer and Neshveyev, working with a definition of pressure for unital nuclear C^*–algebras and the Connes–Narnhofer–Thirring (henceforth abbreviated CNT) entropy, have obtained a variational principle for a class of asymptotically Abelian automorphisms of AF C^*–algebras. We will work within the domain of exact C^*–algebras, replacing the space X by an exact C^*–algebra $\mathcal{A}$ and θ by a contractive completely positive self-map of $\mathcal{A}$ and taking the potential to be a fixed self-adjoint element a of $\mathcal{A}$, and we will establish the variational principle for a class of C^*–dynamical systems which are generally not asymptotically Abelian.

In Sect. 2 we introduce a notion of pressure for a following Voiculescu's approach to topological entropy for unital nuclear C^*–algebras, recently extended to exact C^*–algebras by Brown [7, 29]. Thus our corresponding partition function is computed by means of an optimal approximation, in some sense, of an embedding of $\mathcal{A}$ into some $\mathcal{B}(\mathcal{H})$ via factorizations through finite-dimensional C^*–algebras. Our definition reduces to Walters' pressure when the system $(\mathcal{A}, \theta)$ arises from a topological dynamical system over a compact space, and also to the pressure introduced by Neshveyev and Størmer [18] for unital nuclear C^*–algebras. One advantage of this more general framework is the yield of an immediate proof of the property that pressure decreases when taking C^*–subalgebras, a fact that has been already pointed out by Brown for topological entropy [7].

Among a few other basic properties which easily carry over from classical pressure and Voiculescu–Brown topological entropy or from Neshveyev–Størmer pressure, we establish in Sect. 3 the property of subadditivity in tensor product C^*–algebras, i.e., that the pressure of an element of the form $a \otimes 1 + 1 \otimes b$ with respect to a tensor product map is bounded by the sum of the pressures of a and

b. This fact already implies, in the classical case, that pressure is a subadditive function. We don't know, however, whether this still holds for noncommutative pressure.

   We next approach the variational principle, first focusing on a variational inequality which asserts that that the free energy in a given state is bounded above by the pressure. In the nuclear case the CNT entropy provides one natural candidate for defining the free energy, and indeed the corresponding variational inequality holds [18]. In our setting we substitute the Sauvageot–Thouvenot entropy, which is defined for unital C^*–algebras and reduces to the CNT entropy in the nuclear case [26]. In Sect. 4 we introduce, as an alternative, a measure-theoretic entropy for exact C^*–algebras which adopts the approximation framework of Voiculescu's topological entropy, with the logarithm of the rank of the local finite-dimensional algebra being replaced by the entropy of the induced state on the local algebra (see Choda [8] for the nuclear analogue). We show that this local state approximation entropy reduces to the Kolmogorov–Sinai entropy in the classical case, is a concave function of the invariant state, and majorizes the Sauvageot–Thouvenot entropy. The variational inequality is shown to hold if the free energy is defined via the local state approximation entropy, and as a corollary we obtain the same inequality using the Sauvageot–Thouvenot entropy.

   In Sect. 5 we examine pressure in Cuntz–Krieger algebras $\mathcal{O}_A$ and crossed product C^*–algebras by a single automorphism $\mathcal{A} \rtimes_\alpha \mathbb{Z}$. In the former case we compute the pressure of a self-adjoint element f of the canonical Abelian subalgebra of continuous functions on the underlying Markov subshift with respect to the natural unital completely positive map θ of $\mathcal{O}_A$, with the result that it equals the classical pressure with respect to the shift epimorphism. This fact has the consequence that equilibrium states (i.e., θ–invariant states whose free energy reaches the pressure) exist. In particular, we recover in the case $f = 0$ Boca and Goldstein's computation of the topological entropy of θ [4]. The class of crossed products algebras can be regarded, as far as the variational principle is concerned, as a structurally extreme opposite of that of Cuntz–Krieger algebras. We generalize Brown's result on the constancy of topological entropy, so that if a is a self-adjoint element of $\mathcal{A}$ and u is a unitary in the crossed product implementing α, the pressure of a computed with respect to α in $\mathcal{A}$ or $\mathrm{Ad}\, u$ in $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ is the same.

   Regarding $\mathcal{O}_A$ or $\mathcal{A} \rtimes_\alpha \mathbb{Z}$ as a particular case of the Pimsner C^*–algebra $\mathcal{O}_X$ [20] associated to a finitely generated Hilbert bimodule X over a unital exact C^*–algebra $\mathcal{A}$ leads to the problem of investigating the variational principle in $\mathcal{O}_X$. In Sect. 6 we introduce conditions on X which stress the Cuntz–Krieger behaviour of $\mathcal{O}_X$ rather than the the crossed product character, and we show that under these conditions the variational principle holds with the free energy defined using the Sauvageot–Thouvenot entropy. The dynamics here are defined by a unital completely positive map of $\mathcal{O}_X$ implemented by a basis of the bimodule. Our main assumptions are the following. First we assume that the left action of $\mathcal{A}$ on X is defined diagonally by a finite set of endomorphisms of $\mathcal{A}$. Then we restrict the space of potentials, selecting self-adjoint elements which lie in a "diagonal subalgebra" $\mathcal{D}$ of $\mathcal{O}_X$, which is a noncommutative analogue of the canonical maximal Abelian subalgebra of $\mathcal{O}_A$. Finally we assume that the topological entropy of the defining set of endomorphisms of $\mathcal{A}$ is zero. This is the case if, e.g., $\mathcal{A}$ is an inductive limit of finite-dimensional C^*–algebras which are left invariant by the endomorphisms. This last assumption makes it possible to compute explicitly the pressure of a potential a in $\mathcal{D}$ which commutes with both the images of 1 and a itself under

sufficiently many iterates of the defining endomorphisms. This is in fact the main step which leads to the proof of the variational principle. We also consider a subclass of potentials of $\mathcal{D}$ for which equilibrium states exist.

In the last section we touch on the problem of the relationship between the KMS condition and equilibrium, concentrating on the class of Cuntz–Krieger algebras $\mathcal{O}_A$. To every potential $f \in \mathcal{C}(\Lambda_A)$ we associate a one-parameter automorphism group of $\mathcal{O}_A$, and we show that, if the variation of f is small enough and A is aperiodic, the KMS states with respect to this group are in bijective correspondence with positive eigenvectors of the Banach space adjoint $\mathcal{L}_f^*$ of the Ruelle operator $\mathcal{L}_f$ on $\mathcal{C}(\Lambda_A)$. A classical theorem by Ruelle asserts that if f is Hölder continuous, both $\mathcal{L}_f$ and $\mathcal{L}_f^*$ have unique positive eigenvalues, say h and μ, respectively. This result led Ruelle to a proof of the uniqueness of the equilibrium measure for the shift space, which can be identified with the measure whose Radon–Nykodim derivative with respect to μ is h [23, 5, 30]. We show that, on $\mathcal{O}_A$, μ extends naturally to the unique KMS state at inverse temperature 1 and ν to an equilibrium state of $(\mathcal{O}_A, \theta, f)$.

2 Noncommutative approximation pressure

1. Unital exact C^*–algebras

In this section, unless otherwise stated, $\mathcal{A}$ is a unital exact C^*–algebra, θ is a unital completely positive map of $\mathcal{A}$ and $a \in \mathcal{A}$ is a self-adjoint element. The collection of finite subsets of $\mathcal{A}$ will be denoted by $Pf(\mathcal{A})$. We define the pressure of a by approximation through finite-dimensional C^*–algebras in the following way. Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a faithful unital *–representation on a Hilbert space. Since $\mathcal{A}$ is exact and hence nuclearly embeddable [15, 32], for any finite subset $\Omega \subset \mathcal{A}$ and for any $\delta > 0$ there is a finite-dimensional $\mathcal{B}$ and unital completely positive (henceforth abbreviated as u.c.p.) maps $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ such that $\|(\psi \circ \phi)(x) - \pi(x)\| < \delta$ for all $x \in \Omega$. We denote by $\mathrm{CPA}(\pi, \Omega, \delta)$ the set of all such (ϕ, ψ, B). We emphasize that the maps of $\mathrm{CPA}(\pi, \Omega, \delta)$ are unital. We set

$$\Omega^{(n)} := \Omega \cup \cdots \cup \theta^{n-1}(\Omega),$$

$$a^{(n)} = \sum_{j=0}^{n-1} \theta^j(a).$$

We define the *partition function*

$$Z_{\theta,n}(\pi, a, \Omega, \delta) := \inf\{\mathrm{Tr}\ e^{\phi(a^{(n)})} : (\phi, \psi, B) \in \mathrm{CPA}(\pi, \Omega^{(n)}, \delta)\}$$

where Tr denotes the trace of $\mathcal{B}$ with the normalization $\mathrm{Tr}(e) = 1$ for every minimal projection $e \in \mathcal{B}$. Note that, if $\lambda = \min \mathrm{spec}(a)$, then for any $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\pi, \Omega^{(n)}, \delta)$ we have the inequality

$$\mathrm{Tr}\ e^{\phi(a^{(n)})} \geq e^{\lambda n} \mathrm{rank}(\mathcal{B})$$

and so

$$Z_{\theta,n}(\pi, a, \Omega, \delta) \geq e^{\lambda n} rcp(\pi, \Omega^{(n)}, \delta),$$

where rcp stands for the Voiculescu–Brown δ–rank [7, 29]. In particular,

$$Z_{\theta,n}(\pi, a, \Omega, \delta) > 0.$$

Define

$$P_\theta(\pi, a, \Omega, \delta) := \limsup_n \frac{1}{n} \log Z_{\theta,n}(\pi, a, \Omega, \delta),$$

$$P_\theta(\pi, a, \Omega) = \sup_{\delta > 0} P_\theta(\pi, a, \Omega, \delta),$$

$$P_\theta(\pi, a) = \sup_{\Omega \in Pf(\mathcal{A})} P_\theta(\pi, a, \Omega).$$

We will refer to $P_\theta(\pi, a)$ as the *approximation pressure* (or simply *pressure*) of a (with respect to θ). Note that, referring to the notation of Brown [7] and Voiculescu [29],

$$Z_{\theta,n}(\pi, 0, \Omega) = rcp(\pi, \Omega^{(n)}, \delta),$$

and so

$$P_\theta(\pi, 0, \Omega, \delta) = ht(\pi, \theta, \Omega, \delta),$$

$$P_\theta(\pi, 0, \Omega) = ht(\pi, \theta, \Omega),$$

$$P_\theta(\pi, 0) = ht(\pi, \theta) = \text{the Voiculescu–Brown entropy of } \theta.$$

The first fact that we want to establish is that the partition function, and therefore the pressure, does not depend upon the representation π. This will be done by generalizing arguments of Brown [7] for the entropy.

Proposition 2.1. *If π_1 and π_2 are faithful and unital *–representations of $\mathcal{A}$,*

$$Z_{\theta,n}(\pi_1, a, \Omega, \delta) = Z_{\theta,n}(\pi_2, a, \Omega, \delta),$$

and so

$$P_\theta(\pi_1, a) = P_\theta(\pi_2, a).$$

Proof. Given $\epsilon > 0$, choose $(\phi, \psi, \mathcal{B}) \in \text{CPA}(\pi_1, \Omega^{(n)}, \delta)$ such that

$$\text{Tr } e^{\phi(a^{(n)})} - Z_{\theta,n}(\pi_1, a, \Omega, \delta) < \epsilon.$$

Consider the map $\pi_2 \circ \pi_1^{-1} : \pi_1(\mathcal{A}) \to \mathcal{B}(\mathcal{H}_{\pi_2})$. Apply Arveson's extension theorem [3] to extend this map to a u.c.p. map $T : \mathcal{B}(\mathcal{H}_{\pi_1}) \to \mathcal{B}(\mathcal{H}_{\pi_2})$. Then $(\phi, T \circ \psi, \mathcal{B}) \in \text{CPA}(\pi_2, \Omega^{(n)}, \delta)$ and so we easily obtain $Z_{\theta,n}(\pi_2, a, \Omega, \delta) \leq Z_{\theta,n}(\pi_1, a, \Omega, \delta)$. The opposite inequality follows by exchanging the roles of π_1 and π_2. $\qquad\square$

As a result of this proposition we can avoid specifying the representation π in the partition function as well as in the approximation pressures.

2. Unital nuclear C^*–algebras

Let Ω be a finite subset of $\mathcal{A}$, $\delta > 0$, and $n \in \mathbb{N}$. If $\mathcal{A}$ is a nuclear C^*–algebra, in the definition of pressure it is more natural to replace $\mathrm{CPA}(\pi, \Omega^{(n)}, \delta)$ with the set $\mathrm{CPA}_{\mathrm{nuc}}(\pi, \Omega^{(n)}, \delta)$ of all triples $(\phi, \psi, \mathcal{B})$ where $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{A}$ are u.c.p. maps and $\mathcal{B}$ is a finite–dimensional C^*–algebra such that $\|(\psi \circ \phi)(x) - x\| < \delta$ for all $x \in \Omega^{(n)}$. We thus obtain the corresponding nuclear partition function $Z_{\theta,n}^{\mathrm{nuc}}(a, \Omega, \delta)$, nuclear approximation pressures $P_\theta^{\mathrm{nuc}}(a, \Omega, \delta)$ and $P_\theta^{\mathrm{nuc}}(a, \Omega)$, and nuclear pressure $P_\theta^{\mathrm{nuc}}(a)$, as in [18].

Proposition 2.2. *Let $\mathcal{A}$ be a unital nuclear C^*–algebra faithfully and unitally represented on a Hilbert space $\mathcal{H}$, and let $a \in \mathcal{A}$ be a self-adjoint element. Then for any finite subset $\Omega \subset \mathcal{A}$, $\delta > 0$, and $n \in \mathbb{N}$,*

$$Z_{\theta,n}^{\mathrm{nuc}}(a, \Omega, \delta) = Z_{\theta,n}(a, \Omega, \delta)$$

and so

$$P_\theta^{\mathrm{nuc}}(a) = P_\theta(a).$$

Proof. Our arguments generalize the corresponding arguments of Brown (Prop. 1.4 of [7]) for the Voiculescu–Brown entropy. Fix $\Omega \in Pf(\mathcal{A})$, $\delta > 0$, and $n \in \mathbb{N}$. We first note that $Z_\theta,(a, \Omega, \delta) \leq Z_{\theta,n}^{\mathrm{nuc}}(a, \Omega, \delta)$ since $\mathrm{CPA}_{\mathrm{nuc}}(\Omega^{(n)}, \delta) \subset \mathrm{CPA}(\pi, \Omega^{(n)}, \delta)$. Given $\epsilon > 0$ let $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\pi, \Omega^{(n)}, \delta)$ be such that

$$\mathrm{Tr}\, e^{\phi(a^{(n)})} - Z_{\theta,n}(a, \Omega, \delta) < \epsilon.$$

Choose a triple $(\rho, \sigma, \mathcal{C}) \in \mathrm{CPA}_{\mathrm{nuc}}(\pi, \Omega^{(n)}, \delta)$ and consider a (unital) completely positive extension $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}$ of ρ, which exists by Arveson's extension theorem [3]. Then $(\phi, \sigma \circ \Phi \circ \psi, \mathcal{B}) \in \mathrm{CPA}_{\mathrm{nuc}}(\pi, \Omega^{(n)}, 2\delta)$ and so we easily deduce that $Z^{\mathrm{nuc}}{}_{\theta,n}(a, \Omega, 2\delta) \leq Z_{\theta,n}(a, \Omega, \delta)$. $\square$

3. Not-necessarily-unital exact C^*–algebras

Let $\mathcal{A}$ be an exact C^*–algebra faithfully represented on a Hilbert space $\mathcal{H}$, and $\theta : \mathcal{A} \to \mathcal{A}$ a completely positive contraction. Following Brown's approach to topological entropy for not-necessarily-unital C^*–algebras, we introduce a partition function Z^0 and corresponding approximation pressures P^0 as in the unital case but with respect to the expanded collection $\mathrm{CPA}_0(\pi, \Omega^{(n)}, \delta)$ of triples $(\phi, \psi, \mathcal{B})$ where $\mathcal{B}$ is a finite-dimensional C^*–algebra and $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ are c.p. contractions such that $\|\psi \circ \phi(x) - x\| < \delta$ for all $x \in \Omega^{(n)}$. $P_\theta^0(\pi, a)$ is still independent of the representation π. We would like to thank G. Gong for pointing out why an equality such as the one in the claim in the proof of the following proposition should hold.

Proposition 2.3. *If $\mathcal{A}$ is unital and exact and θ is u.c.p. then $P_\theta^0(a) = P_\theta(a)$.*

Proof. The inequality $P_\theta^0(a) \leq P_\theta(a)$ follows immediately from the definitions.

To establish the reverse inequality, let Ω be a finite subset of the unit ball of $\mathcal{A}$ containing 1, and suppose $0 \leq \delta \leq \frac{1}{4}$. Let $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}_0(\pi, \Omega^{(n)}, \delta)$, and set $b = \phi(1)$. Let p be a spectral projection of b such that $b_1 := bp \geq (1 - \sqrt{\delta})p$ and $b_2 := b(1 - p) < 1 - \sqrt{\delta}$. We claim that $\psi(b_2) < \sqrt{\delta}$. To see this, suppose to the contrary that $\|\psi(b_2)\| \geq \sqrt{\delta}$. Since ψ is contractive we have

$\left\| \psi \left(b_1 + \frac{1}{1-\sqrt{\delta}} b_2 \right) \right\| \leq \left\| b_1 + \frac{1}{1-\sqrt{\delta}} b_2 \right\| \leq 1$. On the other hand, since $\psi(b) \geq 1 - \|\psi \circ \phi(1) - 1\| > 1 - \delta$, the positivity of ψ yields

$$\psi \left(b_1 + \frac{1}{1-\sqrt{\delta}} b_2 \right) = \psi(b_1 + b_2) + \psi \left(\left(\frac{1}{1-\sqrt{\delta}} - 1 \right) b_2 \right)$$

$$> 1 - \delta + \frac{\sqrt{\delta}}{1-\sqrt{\delta}} \psi(b_2),$$

and since $\left\| \frac{\sqrt{\delta}}{1-\sqrt{\delta}} \psi(b_2) \right\| > \delta$ this implies $\left\| \psi \left(b_1 + \frac{1}{1-\sqrt{\delta}} b_2 \right) \right\| > 1$, producing a contradiction and thus establishing the claim.

Observe now that

$$\|\psi(b_1) - 1\| = \|\psi(b) - \psi(b_2) - 1\|$$
$$\leq \|\psi(b) - 1\| + \|\psi(b_2)\|$$
$$< \delta + \sqrt{\delta}$$
$$< 2\sqrt{\delta}.$$

Thus, if p denotes the support projection of b_1, we have $\|\psi(p) - 1\| \leq \|\psi(p - b_1)\| + \|\psi(b_1) - 1\| < 3\sqrt{\delta}$, and so $\|\psi(p)^2 - \psi(p)\| \leq \|\psi(p)(\psi(p) - 1)\| < 3\sqrt{\delta}$. Appealing to Stinespring's theorem [28, 32] we infer that, for all $x \in \mathcal{B}$, $\|\psi(pxp) - \psi(p)\psi(x)\psi(p)\| < 4\sqrt{3}\sqrt[4]{\delta}$ and hence

$$\|\psi(pxp) - \psi(x)\| \leq \|\psi(pxp) - \psi(p)\psi(x)\psi(p)\| + \|\psi(p)\psi(x)\psi(p) - \psi(x)\|$$
$$< 4\sqrt{3}\sqrt[4]{\delta}\|x\| + 8\sqrt{\delta}\|x\|$$
$$< 16\sqrt[4]{\delta}\|x\|.$$

Set $\mathcal{B}' = p\mathcal{B}p$, and define the u.c.p. map $\phi' : \mathcal{A} \to \mathcal{B}'$ by $\phi'(x) = b_1^{-\frac{1}{2}} \phi(x) b_1^{-\frac{1}{2}}$, with b_1 now being considered as an element of $\mathcal{B}'$. Let $\psi' : \mathcal{B}' \to \mathcal{B}(\mathcal{H})$ be the u.c.p. map given by $\psi'(x) = \psi(b_1)^{-\frac{1}{2}} \psi \left(b_1^{\frac{1}{2}} x b_1^{\frac{1}{2}} \right) \psi(b_1)^{-\frac{1}{2}}$. If $x \in \Omega^{(n)}$ then $\|\psi(p\phi(x)p) - x\| \leq \|\psi(p\phi(x)p - \phi(x))\| + \|\psi \circ \phi(x) - x\| < 16\sqrt[4]{\delta} + \delta < 17\sqrt[4]{\delta}$, and so estimating as does Brown in [7] we obtain $\|\psi' \circ \phi'(x) - \psi(p\phi(x)p)\| < 14(17\sqrt[4]{\delta})$, whence

$$\|\psi' \circ \phi'(x) - x\| \leq \|\psi' \circ \phi'(x) - \psi(p\phi(x)p)\| +$$
$$\|\psi(p\phi(x)p - \phi(x))\| + \|\psi \circ \phi(x) - x\|$$
$$< 255\sqrt[4]{\delta}.$$

We therefore have $(\phi', \psi', \mathcal{B}') \in \mathrm{CPA}(\pi, \Omega^{(n)}, 255\sqrt[4]{\delta})$. Also note that

$$\left\| b_1^{-\frac{1}{2}} \phi(a^{(n)}) b_1^{-\frac{1}{2}} - p\phi(a^{(n)})p \right\| \leq \left\| b_1^{-\frac{1}{2}} \phi(a^{(n)}) b_1^{-\frac{1}{2}} - b_1^{-\frac{1}{2}} \phi(a^{(n)})p \right\|$$
$$+ \left\| b_1^{-\frac{1}{2}} \phi(a^{(n)})p - p\phi(a^{(n)})p \right\|$$
$$\leq n\|a\| \left\| b_1^{-\frac{1}{2}} - p \right\| \left(\left\| b_1^{-\frac{1}{2}} \right\| + 1 \right)$$

$$\leq 2n\|a\|\frac{\sqrt{\delta}}{(1-\sqrt{\delta})^2}$$

so that $b_1^{-\frac{1}{2}}\phi(a^{(n)})b_1^{-\frac{1}{2}} \leq p\phi(a^{(n)})p + 2n\|a\|\frac{\sqrt{\delta}}{(1-\sqrt{\delta})^2}$ and hence

$$
\begin{aligned}
\log \mathrm{Tr}_{\mathcal{B}'}e^{\phi'(a^{(n)})} &\leq \log \mathrm{Tr}_{\mathcal{B}'}e^{p\phi(a^{(n)})p+2n\|a\|\frac{\sqrt{\delta}}{(1-\sqrt{\delta})^2}} \\
&= \log \mathrm{Tr}_{\mathcal{B}'}e^{p\phi(a^{(n)})p} + 2n\|a\|\frac{\sqrt{\delta}}{(1-\sqrt{\delta})^2} \\
&\leq \log \mathrm{Tr}_{\mathcal{B}'}pe^{\phi(a^{(n)})}p + 2n\|a\|\frac{\sqrt{\delta}}{(1-\sqrt{\delta})^2} \\
&\leq \log \mathrm{Tr}_{\mathcal{B}}e^{\phi(a^{(n)})} + 2n\|a\|\frac{\sqrt{\delta}}{(1-\sqrt{\delta})^2},
\end{aligned}
$$

with the second last inequality following from Proposition 3.17 of [19]. It follows that

$$P_\theta^0(\mathcal{A},\Omega,255\sqrt[4]{\delta}) \leq P_\theta^0(\mathcal{A},\Omega,\delta) + 2\|a\|\frac{\sqrt{\delta}}{(1-\sqrt{\delta})^2},$$

from which we conclude that $P_\theta^0(a,\Omega) \leq P_\theta(a,\Omega)$. Thus $P_\theta^0(a) \leq P_\theta(a)$. □

3   Main Properties

It is natural to ask which properties of the Voiculescu–Brown topological entropy or the classical pressure carry over to the pressure of a self-adjoint element in a unital exact C^*–algebra. The following result collects some properties inspired from corresponding properties of the classical pressure [31].

Proposition 3.1. *Let a, b be self-adjoint elements of $\mathcal{A}$.*

(a) *If $a \leq b$, $P_\theta(a) \leq P_\theta(b)$;*

(b) *if $\lambda \in \mathbb{R}$, $P_\theta(a + \lambda I) = P_\theta(a) + \lambda$. In particular $P_\theta(\lambda I) = \lambda + ht(\theta)$;*

(c) *$\min spec(a) + ht(\theta) \leq P_\theta(a) \leq \max spec(a) + ht(a)$, so either $P_\theta(a) < \infty$ for all a or $P_\theta(a) = ht(\theta) = \infty$ for all a;*

(d) *if $ht(\theta) < \infty$, $|P_\theta(a) - P_\theta(b)| \leq \|a - b\|$;*

(e) *$P_\theta(ca) \leq cP_\theta(a)$ if $c \geq 1$ and $P_\theta(ca) \geq cP_\theta(a)$ if $c \leq 1$;*

(f) *$|P_\theta(a)| \leq P_\theta(|a|)$.*

Proof. Properties (a) and (d) can be established using the Peierls–Bogoliubov inequality (cf. Cor. 3.15 of [19]) as in the proof of Prop. 2.4 in [18] for the nuclear pressure, while (b) and (e) are immediate from the definition, (c) follows from (a) and (b), and (f) follows from (e) and (a).   □

The following facts are also very easy to check.

Proposition 3.2.

(a) $P_\theta(a) = \frac{1}{r} P_{\theta^r}(a^{(r)})$, *if $r \in \mathbb{N}$;*

(b) *if θ is an automorphism, $P_\theta(a) = P_{\theta^{-1}}(a)$;*

(c) $P_\theta(a + \theta(b) - b) = P_\theta(a)$;

(d) *If θ is an automorphism, $P_\theta(\theta(a)) = P_\theta(a)$.*

Proof. The proofs of Propostion 2.4(v)(ii)in [18] for the nuclear approximation pressure can be adapted to establish (a) and (c), respectively. To establish (b), we need only note that $Z_{\theta,n}(a, \Omega, \delta) = Z_{\theta^{-1},n}(a, \Omega, \delta)$ follows from the observation that $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\pi, \Omega \cup \cdots \cup (\theta^{-1})^{n-1}\Omega, \delta)$ if and only if $(\phi \circ \theta^{-n+1}, \tilde{\theta}^{n-1} \circ \psi, \mathcal{B}) \in \mathrm{CPA}(\pi, \Omega \cup \cdots \cup \theta^{n-1}(\Omega), \delta)$, where $\tilde{\theta} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a u.c.p. extension of $\pi \circ \theta \circ \pi^{-1} : \pi(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$ whose existence is guaranteed by Arveson's Extension Theorem. To show (d), we can take $b = a$ in (c). $\qquad\square$

Next we discuss a few properties of the Voiculescu–Brown entropy which easily carry over to pressure.

Proposition 3.3. *(Monotonicity) Let $\mathcal{A}_0 \subset \mathcal{A}$ be a θ–invariant C^*–subalgebra (i.e., $\theta(\mathcal{A}_0) \subset \mathcal{A}_0$) containing a. Then*

$$P_{\theta \restriction \mathcal{A}_0}(a) \leq P_\theta(a).$$

We also have a Kolmogorov–Sinai-type result.

Proposition 3.4. *Let $\{\Omega_\iota : \iota \in I\}$ be a net of finite subsets of $\mathcal{A}$ such that $\bigcup_{\iota \in I} \bigcup_{j \in \mathbb{N}} \theta^j(\Omega_\iota)$ is total. Then*

$$P_\theta(a) = \lim_\iota P_\theta(a, \Omega_\iota).$$

Proof. It is clear that for $\Omega_1 \subset \Omega_2$, $P_\theta(a, \Omega_1) \leq P_\theta(a, \Omega_2)$, and so the limit on the r.h.s. exists and is bounded by the l.h.s. Let $\Omega \in Pf(\mathcal{A})$ and $\delta > 0$. Consider $\iota \in I$ and $N \in \mathbb{N}$ such that for any $x \in \Omega$ there is $x' = \sum_{r \in F, j \leq N} \lambda_{r,j,x} \theta^j(y_{r,x})$ with F a finite set, $y_{r,x} \in \Omega_\iota$ such that $\|x - x'\| < \delta$. Set $\delta' = \frac{\delta}{(N+1)\mathrm{Card}(\Omega_\iota) \max_{r,j,x} |\lambda_{r,j,x}|}$. For each $n \in \mathbb{N}$ take a triple $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\Omega_\iota^{(n+N+1)}, \delta')$ such that

$$\mathrm{Tr}\ e^{\phi(a^{(n+N+1)})} < 2 Z_{\theta, n+N+1}(a, \Omega_\iota, \delta')$$

One can easily show that $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\Omega^{(n)}, 3\delta)$, and so

$$Z_{\theta,n}(a, \Omega, 3\delta) \leq \mathrm{Tr}\ e^{\phi(a^{(n)})} \leq \mathrm{Tr}\ e^{\phi(a^{(n+N+1)}) + (N+1)\|a\|}$$
$$\leq 2 e^{(N+1)\|a\|} Z_{\theta, n+N+1}(a, \Omega_\iota, \delta'),$$

from which we conclude that $P_\theta(a, \Omega) \leq \lim_\iota P_\theta(a, \Omega_\iota)$.   □

The next proposition gives a weak version of subadditivity in a tensor product C^*–algebra. It also extends the entropy tensor product inequalities from [29] to pressure. Note that the class of exact C^*–algebras is closed under taking minimal tensor products [15].

Proposition 3.5. *Let $\theta_1 : \mathcal{A}_1 \to \mathcal{A}_1$ and $\theta_2 : \mathcal{A}_2 \to \mathcal{A}_2$ be u.c.p. maps and let a_1 and a_2 be self-adjoint elements of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Let $\theta : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ be the (u.c.p.) extension of the map $\theta_1 \otimes \theta_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_1 \otimes \mathcal{A}_2$ on the algebraic tensor product. Then*

$$P_\theta(a_1 \otimes 1 + 1 \otimes a_2) \leq P_{\theta_1}(a_1) + P_{\theta_2}(a_2),$$

Proof. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be faithfully represented on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then $\mathcal{A}_1 \otimes \mathcal{A}_2$ is faithfully represented on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Let $\Omega_1 \in Pf(\mathcal{A}_1)$, $\Omega_2 \in Pf(\mathcal{A}_2)$, and $\delta_1, \delta_2 > 0$. Set $M = \max\{\|x\| : x \in \Omega_1 \cup \Omega_2\}$. Suppose $(\phi_j, \psi_j, \mathcal{B}_j) \in \mathrm{CPA}(\mathcal{A}_j, \Omega_j^{(n)}, \delta_j)$ for $j = 1, 2$. Let $\phi : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \to \mathcal{B}_1 \otimes \mathcal{B}_2$ be the (u.c.p.) extension of the map $\phi_1 \otimes \phi_2 : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{B}_1 \otimes \mathcal{B}_2$. If $x_1 \in \mathcal{A}_1$ and $x_2 \in \mathcal{A}_2$ then

$$
\begin{aligned}
&\|((\psi_1 \otimes \psi_2) \circ (\phi_1 \otimes \phi_2))(x_1 \otimes x_2) - x_1 \otimes x_2\| \\
&\quad = \quad \|(\psi_1 \circ \phi_1)(x_1) \otimes (\psi_2 \circ \phi_2)(x_2) - x_1 \otimes x_2\| \\
&\quad \leq \quad \|(\psi_1 \circ \phi_1)(x_1) - x_1\|\|x_2\| + \|x_1\|\|(\psi_2 \circ \phi_2)(x_2) - x_2\|
\end{aligned}
$$

and so $(\phi, \psi_1 \otimes \psi_2, \mathcal{B}_1 \otimes \mathcal{B}_2) \in \mathrm{CPA}((\Omega_1 \otimes \Omega_2)^{(n)}, M(\delta_1 + \delta_2))$.

Let $(e_k^1)_{k=1}^{\mathrm{rank}(\mathcal{B}_1)}$ and $(e_l^2)_{l=1}^{\mathrm{rank}(\mathcal{B}_2)}$ be maximal sets of pairwise orthogonal minimal spectral projections for $\phi((a_1 \otimes 1)^{(n)})$ and $\phi((1 \otimes a_2)^{(n)})$, respectively. Then $(e_k^1 \otimes e_l^2)_{1 \leq k \leq \mathrm{rank}(\mathcal{B}_1), 1 \leq l \leq \mathrm{rank}(\mathcal{B}_2)}$ is a maximal set of pairwise orthogonal minimal spectral projections for $\phi((a_1 \otimes 1)^{(n)} + (1 \otimes a_2)^{(n)})$, and so

$$
\begin{aligned}
\mathrm{Tr}_{\mathcal{B}_1 \otimes \mathcal{B}_2}\, & e^{\phi\left((a_1 \otimes 1)^{(n)} + (1 \otimes a_2)^{(n)}\right)} \\
&= \quad \sum_{k,l} e^{\mathrm{Tr}_{\mathcal{B}_1 \otimes \mathcal{B}_2}\left((e_k^1 \otimes e_l^2)\phi\left((a_1 \otimes 1)^{(n)} + (1 \otimes a_2)^{(n)}\right)\right)} \\
&= \quad \sum_{k,l} e^{\mathrm{Tr}_{\mathcal{B}_1}\left(e_k^1 \phi_1\left(a_1^{(n)}\right)\right) + \mathrm{Tr}_{\mathcal{B}_2}\left(e_l^2 \phi_2\left(a_2^{(n)}\right)\right)} \\
&= \quad \sum_k e^{\mathrm{Tr}_{\mathcal{B}_1}\left(e_k^1 \phi_1\left(a_1^{(n)}\right)\right)} \sum_l e^{\mathrm{Tr}_{\mathcal{B}_2}\left(e_l^2 \phi_2\left(a_2^{(n)}\right)\right)} \\
&= \quad \mathrm{Tr}_{\mathcal{B}_1}\, e^{\phi_1\left(a_1^{(n)}\right)}\, \mathrm{Tr}_{\mathcal{B}_2}\, e^{\phi_2\left(a_2^{(n)}\right)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
Z_{\theta,n}\,(a_1 \otimes 1 + 1 \otimes a_2, \Omega_1 \otimes \Omega_2, M\,(\delta_1 + \delta_2)) \\
\leq \quad Z_{\theta_1,n}(a_1, \Omega_1, \delta_1)\, Z_{\theta_2,n}(a_2, \Omega_2, \delta_2),
\end{aligned}
$$

and since $\mathcal{A}_1 \otimes \mathcal{A}_2$ is dense in $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ it follows from Prop. 3.4 that

$$P_\theta(a_1 \otimes 1 + 1 \otimes a_2) \leq P_{\theta_1}(a_1) + P_{\theta_2}(a_2).$$

□

If (X, T) is a topological dynamical system over compact metric space X, and a is a real-valued continuous function over X, we denote by $p_T(a)$ the classical topological pressure of a considered by Walters [31].

Proposition 3.6. *Let X be a compact metric space, $T : X \to X$ a continuous function and a a real-valued continuous function on X. Then*

$$P_{\theta_T}(a) = p_T(a)$$

where θ_T is the *–homomorphism of $\mathcal{C}(X)$ defined by $\theta_T(f)(x) = f(Tx)$.

Proof. Since the nuclear and exact approximation pressures agree for nuclear C^*–algebras by Prop. 2.2, we can appeal to Remark 2.3 of [18]. □

Remark. It is natural to ask whether, as in the classical situation, the function $a \to P_\theta(a)$ is convex or subadditive in the case where $ht(\theta)$ is finite. We just note that in the classical situation tensor product subadditivity combined with monotonicity of the classical pressure when passing to closed invariant subspaces implies subadditivity.

4   Entropy and variational inequalities

Our next aim is to establish a variational inequality bounding the free energy in a given state by the pressure. We first introduce a notion of exact–C^*–algebraic entropy with respect to an invariant state which adopts the approximation framework of Voiculescu's topological definition [29], but exercises the entropy of the induced local state instead of the logarithm of the rank of the local algebra (see [8] for the nuclear analogue). The local state approximation entropy yields as a straightforward consequence of its definition the desired variational inequality (Prop. 4.14), and since it majorizes both the Sauvageot–Thouvenot and CNT entropies (Prop. 4.10) the inequality will also hold upon substituting either of the latter as the entropy term in the free energy. To conclude the section we collect some facts about the Sauvageot–Thouvenot entropy which will be needed in Sect. 6.

Let $\mathcal{A}$ be a unital exact C^*–algebra, $\theta : \mathcal{A} \to \mathcal{A}$ a u.c.p. map, and σ a θ–invariant state on $\mathcal{A}$. Let $\mathcal{D}$ be an injective C^*–algebra and $\iota : \mathcal{A} \to \mathcal{D}$ a unital complete order (henceforth abbreviated u.c.o.) embedding. For $\Omega \in Pf(\mathcal{A})$ and $\delta > 0$ we denote by $\text{CPA}(\iota, \Omega, \delta)$ the set of all triples $(\phi, \psi, \mathcal{B})$ where $\mathcal{B}$ is a finite-dimensional C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{D}$ are u.c.p. maps such that $\|(\psi \circ \phi)(x) - \iota(x)\| < \delta$ for all $x \in \Omega$. Since $\mathcal{A}$ is nuclearly embeddable [15], the set $\text{CPA}(\iota, \Omega, \delta)$ is non-empty. Denote by $\mathfrak{E}(\sigma, \iota)$ the set of all states ω on $\mathcal{D}$ which extend the state $\sigma \circ \iota^{-1}$ on $\iota(\mathcal{A})$.

Definition 4.1. *If ω is a state on $\mathcal{D}$, $\Omega \in Pf(\mathcal{A})$, and $\delta > 0$, we define the completely positive δ–entropy*

$$cpe(\iota, \omega, \Omega, \delta) = \inf \{S(\omega \circ \psi) : (\phi, \psi, \mathcal{B}) \in CPA(\iota, \Omega, \delta)\}$$

of Ω with respect to (ι, ω), and for $\omega \in \mathfrak{E}(\sigma, \iota)$ we define the dynamical entropies

$$hm_\sigma(\theta, \iota, \omega, \Omega, \delta) = \limsup_{n \to \infty} \frac{1}{n} cpe(\iota, \omega, \Omega^{(n)}, \delta)$$

$$hm_\sigma(\theta, \iota, \omega, \Omega) = \sup_{\delta > 0} hm_\sigma(\theta, \iota, \omega, \Omega, \delta)$$

$$hm_\sigma(\theta, \iota, \omega) = \sup_{\Omega \in Pf(\mathcal{A})} hm_\sigma(\theta, \iota, \omega, \Omega)$$

$$hm_\sigma(\theta, \iota) = \sup_{\omega \in \mathfrak{E}(\mathcal{D}, \sigma, \iota)} hm_\sigma(\theta, \iota, \omega).$$

We will refer to $hm_\sigma(\theta, \iota)$ as the *local state approximation entropy* of θ.

Proposition 4.2. *If $\iota_1 : \mathcal{A} \to \mathcal{D}_1$ and $\iota_2 : \mathcal{A} \to \mathcal{D}_2$ are u.c.o. embeddings into injective C^*–algebras $\mathcal{D}_1$ and $\mathcal{D}_2$ then*

$$hm_\sigma(\theta, \iota_1) = hm_\sigma(\theta, \iota_2).$$

Proof. Since $\mathcal{D}_1$ is injective we can extend the map $\iota_1 \circ \iota_2^{-1} : \iota_2(\mathcal{A}) \to \mathcal{D}_1$ to a u.c.p. map $\Upsilon : \mathcal{D}_2 \to \mathcal{D}_1$. Let $\omega_1 \in \mathfrak{E}(\sigma, \iota_1)$ and define $\omega_2 \in \mathfrak{E}(\sigma, \iota_2)$ by $\omega_2 = \omega_1 \circ \Upsilon$. Let $\Omega \in Pf(\mathcal{A})$ and $\delta > 0$, and suppose $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\iota_2, \Omega^{(n)}, \delta)$. Then for all $x \in \Omega^{(n)}$,

$$\|((\Upsilon \circ \psi) \circ \phi)(x) - \iota_1(x)\| = \|\Upsilon((\psi \circ \phi)(x) - \iota_2(x))\| \leq \|(\psi \circ \phi)(x) - \iota_2(x)\| < \delta,$$

so that $(\phi, \Upsilon \circ \psi, \mathcal{B}) \in \mathrm{CPA}(\iota_1, \Omega^{(n)}, \delta)$. Since $S(\omega_1 \circ (\Upsilon \circ \psi)) = S(\omega_2 \circ \psi)$, we conclude that

$$cpe(\iota_1, \omega_1, \Omega^{(n)}, \delta) \leq cpe(\iota_2, \omega_2, \Omega^{(n)}, \delta).$$

Thus $hm_\sigma(\theta, \iota_1, \omega_1) \leq hm_\sigma(\theta, \iota_2, \omega_2)$ and so, taking the supremum over $\omega_1 \in \mathfrak{E}(\sigma, \iota_1)$, we obtain $hm_\sigma(\theta, \iota_1) \leq hm_\sigma(\theta, \iota_2)$. The reverse inequality follows by symmetry. $\square$

Definition 4.3. *In view of the above proposition and the fact that $\mathcal{A}$ always admits a u.c.o. embedding into an injective C^*–algebra (consider, for instance, its universal representation), we can define $hm_\sigma(\theta)$ to be $hm_\sigma(\theta, \iota)$ for any u.c.o. embedding $\iota : \mathcal{A} \to \mathcal{D}$ into an injective C^*–algebra $\mathcal{D}$.*

Remark. If $\mathcal{A}$ is nuclear, we can dispense with state extension and define $hm_\sigma^{\mathrm{nuc}}(\theta)$, as does Choda in [8] with different notation, by replacing the logarithm of the rank of the local algebra in Voiculescu's topological definition [29] with the entropy of the induced local state. We can also adapt Voiculescu's AF definition [29] in a similar way to define $hm_\sigma^{\mathrm{AF}}(\theta)$ using the local characterization for AF algebras. Then

$$hm_\sigma(\theta) \leq hm_\sigma^{\mathrm{nuc}}(\theta) \leq hm_\sigma^{\mathrm{AF}}(\theta),$$

with each inequality applying to the appropriate domain of definition.

We show that, as for pressure, the local state approximation entropy can be computed by means of the larger class of contractive c.p. maps. If ω is a state on $\mathcal{D}$, $\Omega \in Pf(\mathcal{A})$, and $\delta >$, we define

$$cpe_0(\iota, \omega, \Omega, \delta) = \inf\{S(\omega \circ \psi) : (\phi, \psi, \mathcal{B}) \in \mathrm{CPA}_0(\iota, \Omega, \delta)\},$$

and we define $hm_\sigma^0(\theta, \iota, \omega, \Omega, \delta)$, etc, in the usual way.

Lemma 4.4. *Let $\mathcal{B}$ be a finite dimensional C^*–algebra and $a, b \in \mathcal{B}$ positive elements with b invertible and $b \leq 0$, and suppose $a \leq \frac{1}{1+\epsilon}$ and $\|b^{-2} - 1\| \leq \epsilon$, for some $\epsilon < 1$. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a concave function which is nonnegative-valued in $[0, 1]$ and increasing in some interval $[0, \alpha]$ with $\epsilon \leq \frac{\alpha}{2}$. Let q be a spectral projection of a such that $qaq \leq \alpha/2$. Then*

$$\mathrm{Tr}_{\mathcal{B}}(f(b^{-1}ab^{-1})) \geq \frac{1}{1+\epsilon}\mathrm{Tr}_{\mathcal{B}}(f(qaq)).$$

Proof. Writing the spectral decompositions $a = \sum_i \nu_j q_j$ and $b^{-1}ab^{-1} = \sum_i \mu_i p_i$, with p_i and q_j minimal projections of $\mathcal{B}$, we have

$$\mu_i = \mathrm{Tr}_{\mathcal{B}}(b^{-1}ab^{-1}p_i) = \sum_j \nu_j \mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i)\frac{\mathrm{Tr}_{\mathcal{B}}(b^{-1}q_jb^{-1}p_i)}{\mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i)},$$

and so by the concavity of f

$$f(\mu_i) \geq \sum_j f(\nu_j \mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i))\frac{\mathrm{Tr}_{\mathcal{B}}(b^{-1}q_jb^{-1}p_i)}{\mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i)}.$$

On the other hand, for all j we have

$$0 \leq \nu_j \mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i) \leq (1+\epsilon)\nu_j \leq 1,$$

and thus, since f is nonnegative $[0, 1]$,

$$f(\mu_i) \geq \sum_{\{j:\nu_j \leq \alpha/2\}} f(\nu_j \mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i))\frac{\mathrm{Tr}_{\mathcal{B}}(b^{-1}q_jb^{-1}p_i)}{\mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i)}.$$

If $\nu_j \leq \alpha/2$ then

$$|\nu_j \mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i) - \nu_j| = \nu_j \mathrm{Tr}_{\mathcal{B}}((b^{-2} - 1)p_i) \leq \epsilon\nu_j \leq \alpha/2,$$

and so $\nu_j \leq \nu_j \mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i) \leq \alpha$. Since f is increasing in $[0, \alpha]$, we have

$$f(\mu_i) \geq \sum_{\{j:\nu_j \leq \alpha/2\}} f(\nu_j)\frac{\mathrm{Tr}_{\mathcal{B}}(b^{-1}q_jb^{-1}p_i)}{\mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i)}.$$

Therefore, summing up over i,

$$\mathrm{Tr}_{\mathcal{B}}(f(b^{-1}ab^{-1})) \geq \sum_i \sum_{\{j:\nu_j \leq \alpha/2\}} f(\nu_j)\frac{\mathrm{Tr}_{\mathcal{B}}(b^{-1}q_jb^{-1}p_i)}{\mathrm{Tr}_{\mathcal{B}}(b^{-2}p_i)}$$

$$\geq \frac{1}{1+\epsilon}\mathrm{Tr}_{\mathcal{B}}(b^{-1}f(qaq)b^{-1})$$

$$\geq \frac{1}{1+\epsilon}\mathrm{Tr}_{\mathcal{B}}(f(qaq)).$$

$\square$

Proposition 4.5. *We have $hm_\sigma^0(\theta, \iota) = hm_\sigma(\theta, \iota)$.*

Proof. Clearly $cpe_0(\iota, \omega, \Omega, \delta) \le cpe(\iota, \omega, \Omega, \delta)$. To show the reverse inequality, let Ω be a subset of the unit ball of $\mathcal{A}$ containing 1 and δ a positive number such that $18\sqrt[4]{\delta} \le \frac{1}{3}$, and, for $n \in \mathbb{N}$, let $(\phi_n, \psi_n, \mathcal{B}_n) \in \mathrm{CPA}_0(\iota, \Omega^{(n)}, \delta)$ be such that $S(\omega \circ \psi_n) < 1 + cpe_0(\iota, \omega, \Omega^{(n)}, \delta)$. We start following the same procedure as in the proof of Prop. 2.3 to obtain a corner $\mathcal{B}_n'$ of $\mathcal{B}_n$ obtained by cutting with a (nonzero) spectral projection p of $\phi_n(1)$ such that $b_1 := p\phi_n(1) \ge (1 - \sqrt{\delta})p$. We shall need the following estimates proven in Prop. 2.3:

$$\left\| b_1^{-\frac{1}{2}} - p \right\| \le \frac{\sqrt{\delta}}{1 - \sqrt{\delta}},$$
$$\| \psi_n(pxp) - \psi_n(x) \| < 16\sqrt[4]{\delta}\|x\|,$$
$$\| 1 - \psi_n(b_1) \| < 2\sqrt{\delta}.$$

We first define $\phi_n' : \mathcal{A} \to \mathcal{B}_n$ and $\psi_n' : \mathcal{B}_n \to \mathcal{B}(\mathcal{H})$ by

$$\phi_n'(t) := b_1^{-\frac{1}{2}}\phi_n(t)b_1^{-\frac{1}{2}} + \gamma(t)(1 - p),$$

where γ is any state of $\mathcal{A}$, and

$$\psi_n'(t) := \psi_n\big((b_1^{\frac{1}{2}} + 1 - p)t(b_1^{\frac{1}{2}} + 1 - p) \big).$$

Note that ϕ'_n is now unital, and

$$\| \psi_n'\phi_n'(t) - \iota(t) \| \le \| \psi_n(p\phi_n(t)p) - \iota(t) \| + \|t\|\|\psi_n(1 - p)\|$$
$$\le 32\sqrt[4]{\delta}\|t\| + \|\psi_n \circ \phi_n(t) - \iota(t)\|$$

so that $(\phi_n', \psi_n', \mathcal{B}_n) \in \mathrm{CPA}_0(\Omega^{(n)}, 33\sqrt[4]{\delta})$. We next fix ψ_n' in order to obtain a unital map. Define $\phi_n'' : \mathcal{A} \to \mathcal{B}_n \oplus \mathbb{C}$ and $\psi_n'' : \mathcal{B}_n \oplus \mathbb{C} \to \mathcal{B}(\mathcal{H})$ by

$$\phi_n''(t) = \phi'_n(t) \oplus \gamma(t),$$
$$\psi_n''(t \oplus \lambda) = \psi'_n(t) + \lambda(1 - \psi_n(b_1 + 1 - p)).$$

Note that, for $t \in \mathcal{A}$,

$$\| \psi_n'' \circ \phi_n''(t) - \iota(t) \| \le \| \psi_n' \circ \phi'_n(t) - \iota(t) \| + \|t\|\|1 - \psi_n(b_1)\| + \|t\|\|\psi_n(1 - p)\|,$$

and so $(\phi_n'', \psi_n'', \mathcal{B}_n \oplus \mathbb{C}) \in \mathrm{CPA}(\iota, \Omega^{(n)}, 51\sqrt[4]{\delta})$. We next estimate $\limsup_n \frac{1}{n}S(\omega \circ \psi_n'')$. If $A \in \mathcal{B}_n$ denotes the density matrix of $\omega \circ \psi_n$, $B := (b_1^{\frac{1}{2}} + 1 - p)A(b_1^{\frac{1}{2}} + 1 - p) \oplus \alpha_n$ is the density matrix of $\omega \circ \psi_n''$, where $\alpha_n = \omega(1 - \psi_n(b_1 + 1 - p))$. We claim that

$$\limsup_n \frac{1}{n}S(\omega \circ \psi_n'') = \limsup_n \frac{1}{n}\mathrm{Tr}_{\mathcal{B}_n}\eta\big(q(b_1^{\frac{1}{2}} + 1 - p)A(b_1^{\frac{1}{2}} + 1 - p)q\big),$$

where $\eta(x) = -x \log x$ and q is a spectral projection of $\big(b_1^{\frac{1}{2}} + 1 - p\big)A\big(b_1^{\frac{1}{2}} + 1 - p\big)$ such that $q\big(b_1^{\frac{1}{2}} + 1 - p\big)A\big(b_1^{\frac{1}{2}} + 1 - p\big) \leq \frac{1}{3}$ and $(1-q)\big(b_1^{\frac{1}{2}} + 1 - p\big)A\big(b_1^{\frac{1}{2}} + 1 - p\big) \geq \frac{1}{3}(1-q)$. To establish the claim, let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $\big(b_1{}^{\frac{1}{2}} + 1 - p\big)A\big(b_1{}^{\frac{1}{2}} + 1 - p\big)$; then $(\lambda_1, \ldots, \lambda_N, \alpha_n)$ are the eigenvalues of B. Then

$$
\begin{aligned}
\mathrm{Tr}_{\mathcal{B}_n}\big(\eta\big(q\big(b_1^{\frac{1}{2}} & + 1 - p\big)A\big(b_1^{\frac{1}{2}} + 1 - p\big)q\big)\big) \\
&= -\sum_{\{i:\lambda_i \leq \frac{1}{3}\}} \lambda_i \log \lambda_i \\
&\leq -\sum_{\{i:\lambda_i \leq \frac{1}{3}\}} \lambda_i \log \lambda_i - \sum_{\{i:\lambda_i > \frac{1}{3}\}} \lambda_i \log \lambda_i - \alpha_n \log \alpha_n \\
&= S(\omega \circ \psi_n'') \\
&\leq \mathrm{Tr}_{\mathcal{B}_n}\big(\eta\big(q\big(b_1^{\frac{1}{2}} + 1\big)A\big(b_1^{\frac{1}{2}} + 1\big)q\big)\big) + \log 3 - \alpha_n \log(\alpha_n).
\end{aligned}
$$

Since $0 \leq \alpha_n \leq 1$ for all n, we have $0 \leq -\alpha_n \log \alpha_n < 1$, and therefore the claim follows by dividing by n and taking the $\limsup_n$. Applying the previous lemma to the matrices

$$
a = \frac{1}{1+\epsilon}\big(b_1^{\frac{1}{2}} + 1 - p\big)A\big(b_1^{\frac{1}{2}} + 1 - p\big),
$$
$$
b = b_1^{\frac{1}{2}} + 1 - p,
$$

the function $f = \eta$, and $\epsilon = \frac{\sqrt{\delta}}{1-\sqrt{\delta}}$, we see that

$$
(1+\epsilon)\mathrm{Tr}_{\mathcal{B}_n}\Big(\eta\Big(\frac{1}{1+\epsilon}A\Big)\Big) \geq \mathrm{Tr}_{\mathcal{B}_n}\Big(\eta\Big(\frac{1}{1+\epsilon}q\big(b_1^{\frac{1}{2}} + 1\big)A\big(b_1^{\frac{1}{2}} + 1\big)q\Big)\Big),
$$

and so $\lim_{\delta \to 0} \limsup_n \frac{1}{n}S(\omega \circ \psi_n'') \leq \lim_{\delta \to 0} \limsup_n \frac{1}{n}S(\omega \circ \psi_n)$. We conclude that $hm_\sigma(\theta, \iota, \omega, \Omega) \leq hm_\sigma^0(\theta, \iota, \omega, \Omega)$, completing the proof. $\qquad\qquad \square$

We discuss some basic properties of the local state approximation entropy.

Proposition 4.6. *Let $\mathcal{C}$ be a unital θ–invariant C^*–subalgebra of $\mathcal{A}$ and $E : \mathcal{A} \to \mathcal{C}$ a conditional expectation. If σ is a θ–invariant and E–invariant state of $\mathcal{A}$,*

$$
hm_{\sigma \restriction \mathcal{C}}(\theta \restriction \mathcal{C}) \leq hm_\sigma(\theta).
$$

Proof Let $\iota : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be the universal representation of $\mathcal{A}$, $\omega \in \mathfrak{E}(\sigma \restriction \mathcal{C}, \iota \restriction \mathcal{C})$, $\Omega \in Pf(\mathcal{C})$ and $\delta > 0$. Extend $\iota \circ E \circ \iota^{-1} : \iota(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$ to a u.c.p. map $\tilde{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, so $\iota \circ E = \tilde{E} \circ \iota$ on $\mathcal{A}$. Thus, by E–invariance of σ, $\omega \circ \tilde{E} \in \mathfrak{E}(\sigma, \iota)$. Now, if $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\iota \restriction \mathcal{C}, \Omega^{(n)}, \delta)$ then $(\phi, \tilde{E} \circ \psi, \mathcal{B}) \in \mathrm{CPA}(\iota \restriction \mathcal{C}, \Omega^{(n)}, \delta)$ as well, so

$$
cpe(\iota \restriction \mathcal{C}, \omega, \Omega, \delta) \leq \inf\{S(\omega \circ \tilde{E} \circ \psi) : (\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\iota \restriction \mathcal{C}, \Omega^{(n)}, \delta)\} \leq
$$

$$
\inf\{S(\omega \circ \tilde{E} \circ \psi) : (\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\iota, \Omega^{(n)}, \delta)\} = cpe(\iota, \omega \circ \tilde{E}, \Omega^{(n)}, \delta)
$$

which implies

$$hm_{\sigma \restriction \mathcal{C}}(\theta \restriction \mathcal{C}, \iota \restriction \mathcal{C}, \omega, \Omega, \delta) \leq hm_\sigma(\theta, \iota),$$

and the proof is complete. $\square$

Proposition 4.7. *If $k \in \mathbb{N}$ then*

$$hm_\sigma(\theta^k) = k\, hm_\sigma(\theta).$$

Proof. Since $cpe(\iota, \omega, \Omega, \delta)$ is defined by taking an infimum over $\mathrm{CPA}(\iota, \Omega, \delta)$, the second half of the proof of Prop. 1.3 in [29] can be immediately adapted to our situation to establish the equality. Explicitly, we have

$$hm_\sigma(\theta^k, \iota, \omega, \Omega, \delta) \leq k\, hm_\sigma(\theta, \iota, \omega, \Omega, \delta)$$

because

$$\mathrm{CPA}\Big(\iota, \bigcup_{j=0}^{n-1} \theta^{jk}(\Omega), \delta\Big) \supset \mathrm{CPA}\Big(\iota, \bigcup_{j=0}^{k(n-1)} \theta^j(\Omega), \delta\Big)$$

for all $n \in \mathbb{N}$, while the inequality

$$hm_\sigma\Big(\theta^k, \iota, \omega, \bigcup_{j=0}^{k-1} \theta^j(\Omega), \delta\Big) \geq k\, hm_\sigma(\theta, \iota, \omega, \Omega, \delta)$$

follows from the observation that

$$\mathrm{CPA}\Big(\iota, \bigcup_{i=0}^{\lfloor \frac{n}{k} \rfloor} \theta^{ik}\Big(\bigcup_{j=0}^{k-1} \theta^j(\Omega)\Big), \delta\Big) \subset \mathrm{CPA}\Big(\iota, \bigcup_{j=0}^{n-1} \theta^j(\Omega), \delta\Big)$$

for all $n \in \mathbb{N}$, whence the proposition follows by taking the supremum over all $\Omega \in Pf(A)$, $\delta > 0$, and $\omega \in \mathfrak{E}(\sigma, \iota)$ and applying Prop. 4.2. $\square$

The proof of Prop. 3.4 can be adapted to establish the following Kolmogorov–Sinai-type result.

Proposition 4.8. *If $\iota : \mathcal{A} \to \mathcal{D}$ is a u.c.o. embedding in an injective C^*–algebra $\mathcal{D}$, $\{\Omega_\lambda\}_{\lambda \in I}$ is a net of elements of $Pf(\mathcal{A})$ such that $\bigcup_{\lambda \in I} \bigcup_{j \in \mathbb{N}} \theta^j(\Omega_\iota)$ is total in $\mathcal{A}$ then*

$$hm_\sigma(\theta) = \lim_\lambda \sup_{\omega \in \mathfrak{E}(D, \sigma, \iota)} hm_\sigma(\theta, \iota, \omega, \Omega_\lambda).$$

We next compare $hm_\sigma(\theta)$ with the Sauvageot–Thouvenot entropy $h_\sigma(\theta)$. We recall from [26] the notion of a stationary coupling of $(\mathcal{A}, \theta, \sigma)$ with a unital commutative dynamical system $(\mathcal{C}, \varsigma, \mu)$. Since Sauvageot and Thouvenot treat the case in which θ is an automorphism, they assume that ς is an automorphism as well. Since our θ is a u.c.p. map, we will assume, more naturally, that ς is a *–homomorphism. A *stationary coupling* is a $\theta \otimes \varsigma$–invariant state λ on $\mathcal{A} \otimes \mathcal{C}$ such that

$\lambda(a \otimes 1) = \sigma(a)$, $\lambda(1 \otimes c) = \mu(c)$ for all $a \in \mathcal{A}$ and $c \in \mathcal{C}$. The Sauvageot–Thouvenot entropy $h_\sigma(\theta)$ is the supremum of the quantities

$$h'(\mathcal{P}, \lambda) := H_\mu\Big(\mathcal{P} \mid \bigvee_{k=0}^{\infty} \varsigma^k(\mathcal{P})\Big) - H_\mu(\mathcal{P}) + \sum_{p \in \mathcal{P}} \mu(p) S(\sigma, \sigma_p)$$

where $\mathcal{P}$ ranges over all finite partitions of $\mathcal{C}$ into projections and λ over all stationary couplings, with σ_p denoting the state $x \mapsto \frac{1}{\mu(p)}\lambda(x \otimes p)$ on $\mathcal{A}$ and $S(\cdot, \cdot)$ denoting Araki's quantum relative entropy [2]. Setting $\mathcal{P}^- = \bigvee_{k=1}^{n} \varsigma^{-k}\mathcal{P}$, Sauvageot and Thouvenot show that $h_\sigma(\theta)$ may be equivalently defined as the supremum over the same set of $\mathcal{P}$ and λ of the quantity

$$h(\mathcal{P}, \lambda) := H_\mu\Big(\mathcal{P} \mid \bigvee_{k=0}^{\infty} \varsigma^k(\mathcal{P})\Big) - H_\lambda(\mathcal{P} \mid \mathcal{A} \otimes \mathcal{P}^-),$$

where $H_\lambda(\cdot \mid \cdot)$ denotes the conditional entropy as defined in Sect. 2 of [26], here with respect to the stationary coupling (also denoted for notational simplicity and consistency with [26] by λ) of $(\mathcal{A} \otimes \mathcal{C}, \theta \otimes \varsigma, \lambda)$ with $(\mathcal{C}, \varsigma, \mu)$ defined by composing λ with $id_{\mathcal{A}} \otimes S$, where $S : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ acts by restricting functions to the diagonal. We shall find it convenient to use the following equivalent expression for $h(\mathcal{P}, \lambda)$ (cf. Prop. 3.3 of [26]) which involves the mutual entropy of λ with respect to a partition $\mathcal{Q}$ of $\mathcal{C}$ into projections as defined in [26] by

$$\varepsilon_\lambda(\mathcal{A}, \mathcal{Q}) = \sum_{q \in \mathcal{Q}} \mu(q) S(\sigma, \sigma_q).$$

Lemma 4.9. *If λ is a stationary coupling of $(\mathcal{A}, \theta, \sigma)$ with the unital commutative system $(\mathcal{C}, \varsigma, \mu)$ and $\mathcal{P}$ is a finite partition of $\mathcal{C}$ into projections then*

$$h(\mathcal{P}, \lambda) = \lim_{n \to \infty} \frac{1}{n} \varepsilon_\lambda\Big(\mathcal{A}, \bigvee_{k=1}^{n} \varsigma^k(\mathcal{P})\Big).$$

Proof. As described above and in the paragraph preceding Lemma 2.2 in [26], the stationary coupling λ defines a stationary coupling (denoted also by λ) of $(\mathcal{A} \otimes \mathcal{C}, \theta \otimes \varsigma, \lambda)$ with $(\mathcal{C}, \varsigma, \mu)$ via the map from $\mathcal{C} \otimes \mathcal{C}$ to $\mathcal{C}$ which restricts functions to the diagonal. We may also similarly define a stationary coupling (again denoted by λ) of $(\mathcal{A} \otimes \mathcal{C} \otimes \mathcal{C}, \theta \otimes \varsigma \otimes \varsigma, \lambda)$ with $(\mathcal{C}, \varsigma, \mu)$ using the same map from $\mathcal{C} \otimes \mathcal{C}$ to $\mathcal{C}$. For each $n \geq 2$ we then have by Lemma 2.2 of [26]

$$H_\lambda\Big(\bigvee_{k=0}^{n-1} \varsigma^k\mathcal{P} \mid \mathcal{A} \otimes \mathcal{P}^-\Big) = H_\lambda\Big(\bigvee_{k=0}^{n-2} \varsigma^k\mathcal{P} \mid \mathcal{A} \otimes \mathcal{P}^-\Big) + H_\lambda\Big(\varsigma^{n-1}\mathcal{P} \mid \mathcal{A} \otimes \mathcal{P}^- \otimes \bigvee_{k=0}^{n-2} \varsigma^k\mathcal{P}\Big),$$

and since

$$H_\lambda\Big(\varsigma^{n-1}\mathcal{P} \mid \mathcal{A} \otimes \mathcal{P}^- \otimes \bigvee_{k=0}^{n-2} \varsigma^k\mathcal{P}\Big) = H_\lambda\Big(\varsigma^{n-1}\mathcal{P} \mid \mathcal{A} \otimes \bigvee_{k=-\infty}^{n-2} \varsigma^k\mathcal{P}\Big)$$

$$= H_\lambda(\mathcal{P} \,|\, \mathcal{A} \otimes \mathcal{P}^-)$$

this leads inductively to

$$H_\lambda\Big(\bigvee_{k=0}^{n-1} \varsigma^k \mathcal{P} \,\big|\, \mathcal{A} \otimes \mathcal{P}^- \Big) = (n+1) H_\lambda(\mathcal{P} \,|\, \mathcal{A} \otimes \mathcal{P}^-).$$

Noting now that another application of Lemma 2.2 of [26] yields

$$H_\lambda\Big(\bigvee_{k=0}^{n-1} \varsigma^k \mathcal{P} \,\big|\, \mathcal{A} \otimes \mathcal{P}^- \Big) = H_\lambda\Big(\bigvee_{k=1}^{n-1} \varsigma^k \mathcal{P} \,\big|\, \mathcal{A} \Big) - H_\lambda(\mathcal{P} \,|\, \mathcal{A}),$$

we obtain

$$H_\lambda\Big(\bigvee_{k=1}^{n} \varsigma^k \mathcal{P} \,\big|\, \mathcal{A} \Big) = H_\lambda(\mathcal{P} \,|\, \mathcal{A}) + (n+1) H_\lambda(\mathcal{P} \,|\, \mathcal{A} \otimes \mathcal{P}^-).$$

Dividing by n and taking the limit as n tends to infinity yields

$$H_\lambda(\mathcal{P} \,|\, \mathcal{A} \otimes \mathcal{P}^-) = \lim_{n \to \infty} \frac{1}{n} H_\lambda\Big(\bigvee_{k=1}^{n} \varsigma^k \mathcal{P} \,\big|\, \mathcal{A} \Big).$$

Since

$$H_\mu(\mathcal{P} \,|\, \mathcal{P}^-) = \lim_{n \to \infty} \frac{1}{n} H_\mu\Big(\bigvee_{k=1}^{n} \varsigma^k \mathcal{P} \Big)$$

from the classical theory, we conclude that

$$h'(\mathcal{P}, \lambda) = H_\mu(\mathcal{P} \,|\, \mathcal{P}^-) - H_\lambda(\mathcal{P} \,|\, \mathcal{A} \otimes \mathcal{P}^-)$$

$$= \lim_{n \to \infty} \frac{1}{n} \Big[H_\mu\Big(\bigvee_{k=1}^{n} \varsigma^k \mathcal{P} \Big) - H_\lambda\Big(\bigvee_{k=1}^{n} \varsigma^k \mathcal{P} \,\big|\, \mathcal{A} \Big) \Big]$$

$$= \lim_{n \to \infty} \frac{1}{n} \varepsilon_\lambda\Big(\mathcal{A}, \bigvee_{k=1}^{n} \varsigma^k(\mathcal{P}) \Big).$$

$\square$

Proposition 4.10. *If $\theta : \mathcal{A} \to \mathcal{A}$ is a u.c.p. map and σ is a θ–invariant state on $\mathcal{A}$, then*

$$hm_\sigma(\theta) \geq h_\sigma(\theta).$$

Proof. Let $\iota : \mathcal{A} \to \mathcal{D}$ be a u.c.o. embedding into an injective C^*–algebra $\mathcal{D}$. Suppose λ is a stationary coupling of $(\mathcal{A}, \theta, \sigma)$ with $(\mathcal{C}, \varsigma, \mu)$, with μ assumed to be faithful. Extend the state $\lambda \circ (\iota^{-1} \otimes id)$ on $\iota(\mathcal{A}) \otimes \mathcal{C}$ to a state $\tilde\lambda$ on $\mathcal{D} \otimes \mathcal{C}$. Suppose $\mathcal{P}$ is a finite partition of projections in $\mathcal{C}$. For each $n \in \mathbb{N}$ and $p \in \bigvee_{k=1}^{n} \varsigma^k(\mathcal{P})$, let σ_p be the state on $\mathcal{A}$ defined by $x \mapsto \mu(p)^{-1} \lambda(x \otimes p)$ and ω_p the state on $\mathcal{D}$ defined by $y \mapsto \mu(p)^{-1} \tilde\lambda(y \otimes p)$. Note that ω_p extends the state $\sigma_p \circ \iota^{-1}$ on $\iota(\mathcal{A})$. Let ω be the state

on $\mathcal{D}$ given by the convex combination $\sum_{p \in \varsigma(\mathcal{P})} \mu(p)\omega_p$ (which is equal to $\sum_{p \in \bigvee_{k=1}^{n} \varsigma^k(\mathcal{P})} \mu(p)\omega_p$ for any $n \in \mathbb{N}$).

For every $n \in \mathbb{N}$, $\Omega \in Pf(\mathcal{A})$, and $\delta > 0$ choose

$$\left(\phi_{(\Omega,\delta),n}, \psi_{(\Omega,\delta),n}, \mathcal{B}_{(\Omega,\delta),n}\right) \in \mathrm{CPA}(\iota, \Omega^{(n)}, \delta)$$

such that

$$hm_\sigma(\theta, \iota, \omega, \Omega, \delta) = \limsup_{n \to \infty} \frac{1}{n} S\left(\omega \circ \psi_{(\Omega,\delta),n}\right).$$

Set $\Gamma = Pf(\mathcal{A}) \times \mathbb{R}_{>0}$. For each $n \in \mathbb{N}$, $\{\psi_{\gamma,n} \circ \phi_{\gamma,n}\}_{\gamma \in \Gamma}$ is a net converging pointwise in norm to ι, so that $\{\omega \circ \psi_{\gamma,n} \circ \phi_{\gamma,n}\}_{\gamma \in \Gamma}$ converges weak* to σ and, for all $p \in \bigvee_{k=1}^{n} -\varsigma^k(\mathcal{P})$, $\{\omega_p \circ \psi_{\gamma,n} \circ \phi_{\gamma,n}\}_{\gamma \in \Gamma}$ converges weak* to σ_p. The weak* lower semicontinuity of the relative entropy $S(\cdot, \cdot)$ and the weak* compactness of the state space of $\mathcal{A}$ then yields a $\gamma_0 = (\Omega_0, \delta_0) \in \Gamma$ such that, for all $n \in \mathbb{N}$ and $p \in \bigvee_{k=1}^{n} \varsigma^k(\mathcal{P})$,

$$S(\sigma, \sigma_p) < S\left(\omega \circ \psi_{\gamma_0,n} \circ \phi_{\gamma_0,n}, \omega_p \circ \psi_{\gamma_0,n} \circ \phi_{\gamma_0,n}\right) + 1.$$

Since

$$S\left(\omega \circ \psi_{\gamma_0,n} \circ \phi_{\gamma_0,n}, \omega_p \circ \psi_{\gamma_0,n} \circ \phi_{\gamma_0,n}\right) \leq S(\omega \circ \psi_{\gamma_0,n}, \omega_p \circ \psi_{\gamma_0,n})$$

by the monotonicity of $S(\cdot, \cdot)$, we therefore have

$$\begin{aligned}
h(\mathcal{P}, \lambda) &= \lim_{n \to \infty} \frac{1}{n} \varepsilon_\lambda\left(\mathcal{A}, \bigvee_{k=1}^{n} \varsigma^k(\mathcal{P})\right) \\
&= \lim_{n \to \infty} \frac{1}{n} \sum_{p \in \bigvee_{k=1}^{n} \varsigma^k(\mathcal{P})} \mu(p) S(\sigma, \sigma_p) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{p \in \bigvee_{k=1}^{n} \varsigma^k(\mathcal{P})} \mu(p) S(\omega \circ \psi_{\gamma_0,n}, \omega_p \circ \psi_{\gamma_0,n}) \\
&\leq \limsup_{n \to \infty} \frac{1}{n} S(\omega \circ \psi_{\gamma_0,n}) \\
&= hm_\sigma(\theta, \iota, \omega, \Omega_0, \delta_0).
\end{aligned}$$

Taking the supremum over all stationary couplings λ and finite partitions $\mathcal{P}$, we obtain the proposition. $\qquad \square$

Next we show that the local state approximation entropy agrees with the Kolmogorov–Sinai entropy in the commutative case. For an open cover $\mathcal{U}$ of a topological space X we denote by $\mathfrak{S}(\mathcal{U})$ the set of all $x \in X$ which are contained in only one member of $\mathcal{U}$.

Lemma 4.11. *Let μ be a measure on a compact Hausdorff space X. If $\mathcal{U} = \{U_1, \ldots, U_m\}$ is a finite open cover of X, then for every $\epsilon > 0$ there is a open refinement $\mathcal{V} = \{V_1, \ldots V_m\}$ of $\mathcal{U}$ such that there are closed sets $H_i \subset V_i$ for $i = 1, \ldots, m$ such that $\bigcup_{i=1}^{m} H_i \subset \mathfrak{S}(\mathcal{V})$ and $\mu(X \setminus \bigcup_{i=1}^{m} H_i) < \epsilon$.*

Proof. Let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be a finite open cover of X and $\epsilon > 0$. Set $V_1 = U_1$. Let $G_1 \subset U_1$ be a closed set such that $\mu(V_1 \setminus G_1) < \frac{\epsilon}{m^2}$ and set $V_2 = U_2 \cap (X \setminus G_1)$. We continue inductively for $k =$

$3, \ldots, m$ so that at the kth stage we choose a closed set $G_{k-1} \subset V_{k-1}$ such that $\mu(V_{k-1} \setminus G_{k-1}) < \frac{\epsilon}{m^2}$ and set $V_k = U_k \cap (X \setminus \bigcup_{j=1}^{k-1} G_j)$.

Put $H_1 = G_1$ and, for $i = 2, \ldots, m$, $H_i = G_i \setminus (V_1 \cup \cdots \cup V_{i-1})$. Then $\bigcup_{i=1}^{m} H_i \subset \mathfrak{S}(\mathcal{V})$, and since $G_1, \ldots, G_m$ are pairwise disjoint so are $H_1, \ldots, H_m$. Furthermore, for each $i = 1, \ldots, m$ we have

$$\mu(V_i \setminus H_i) \leq \sum_{j=1}^{i} \mu(V_j \setminus G_j) < i\frac{\epsilon}{m^2} \leq \frac{\epsilon}{m},$$

so that

$$\mu\left(X \setminus \bigcup_{1 \leq k \leq m} H_k \right) \leq \mu\left(\bigcup_{1 \leq k \leq m} (V_k \setminus H_k) \right) \leq \sum_{k=1}^{m} \mu(V_k \setminus H_k) < m\frac{\epsilon}{m} = \epsilon,$$

as required. $\square$

Proposition 4.12. *Let $T : X \to X$ be a homeomorphism of a compact metric space and μ a T-invariant measure on X. If θ_T is the automorphism of $C(X)$ induced by T and σ denotes the state on $C(X)$ defined by μ, then*

$$h_\mu(T) = hm_\sigma(\theta_T),$$

where $h_\mu(T)$ is the Kolmogorov-Sinai entropy of T.

Proof. Since the local state approximation entropy is bounded below by the Sauvageot–Thouvenot entropy (Prop. 4.10) and the latter agrees with the Kolmogorov-Sinai entropy in the commutative case, we have $h_\mu(T) \leq hm_\sigma(\theta_T)$.

To establish the reverse inequality, let $\iota : C(X) \to C(X)^{**}$ be the natural embedding. Note that, since $C(X)$ is nuclear, $C(X)^{**}$ is injective [14]. Suppose $\omega \in \mathfrak{E}(\sigma, \iota)$, $\Omega \in Pf(C(X))$ and $\delta > 0$. Let $\mathcal{U}$ be an open cover of X such that if $U \in \mathcal{U}$ and $x, y \in U$ then $|f(x) - f(y)| \leq \delta$ for all $f \in \Omega$. Writing $\mathcal{U} = \{U_1, \ldots, U_r\}$, let $\mathcal{P}$ be the Borel partition $\{U_i \setminus \bigcup_{j=1}^{i-1} U_j : 1 \leq i \leq r\}$ refining $\mathcal{U}$. Fix $n \in \mathbb{N}$. Note that if $U \in \bigvee_{j=0}^{n-1} T^j(\mathcal{U})$ and $x, y, \in U$ then $|f(x) - f(y)| \leq \delta$ for all $f \in \Omega^{(n)}$. Let $\epsilon > 0$ be small enough so that if $0 \leq a, b \leq 1$ and $|a - b| < \epsilon$ then $|a \log a - b \log b| < r^{-n}$. By the lemma there is a refinement $\mathcal{V} = \{V_1, \ldots V_m\}$ of $\bigvee_{j=0}^{n-1} T^j(\mathcal{U})$ such that there are pairwise disjoint closed sets $H_i \subset V_i$ for $i = 1, \ldots, m$ such that $\bigcup_{i=1}^{m} H_i \subset \mathfrak{S}(\mathcal{U})$ and $\mu(X \setminus \bigcup_{i=1}^{m} H_i) < \epsilon$. Let $\Xi = \{\chi_1, \ldots, \chi_m\}$ be a partition of unity subordinate to $\mathcal{V}$ and $X_n = \{x_1, \ldots, x_m\}$ a finite subset of X such that $x_i \in H_i$ for each $i = 1, \ldots, m$. Defining $\phi_n : C(X) \to C(X_n)$ by $f \mapsto f \upharpoonright X_n$ and $\psi_n : C(X_n) \to C(X)^{**}$ by $g \mapsto \sum_{1 \leq i \leq m} g(x_i)\iota \circ \chi_i$. Since Ξ is subordinate to $\bigvee_{j=0}^{n-1} T^j(\mathcal{U})$, for all $f \in \Omega^{(n)}$ and $x \in X$ we have

$$|((\iota^{-1} \circ \psi_n \circ \phi_n)(f))(x) - f(x)| \leq \sum_{1 \leq i \leq m} \chi_i(x)|f(x_i) - f(x)|$$

$$= \sum_{\{i \,:\, x \in V_i\}} \chi_i(x)|f(x_i) - f(x)|$$

$$< \delta$$

and hence

$$|(\psi_n \circ \phi_n)(f) - \iota(f)| < \delta,$$

so that $(\phi_n, \psi_n, C(X_n)) \in \mathrm{CPA}(\iota, \Omega^{(n)}, \delta)$.

Now for each $i = 1, \ldots, m$, we have

$$|\sigma(\chi_i) - \mu(H_i)| \leq \mu(V_i \setminus H_i) < \epsilon$$

since $\mu(X \setminus \bigcup_{1 \leq i \leq m} H_i) < \epsilon$ and V_i does not intersect H_j for $j = 1, \ldots, m, j \neq i$. Thus, since $m \leq r^n$, our choice of ϵ yields

$$\begin{aligned}
S(\omega \circ \psi_n) &= -\sum_{i=1}^{m} \sigma(\chi_i) \log \sigma(\chi_i) \\
&\leq -\sum_{i=1}^{m} \mu(H_i) \log \mu(H_i) + 1.
\end{aligned}$$

Setting $K = \bigcup_{i=1}^{m} H_i$ we have $|\mu(P \cap K) - \mu(P)| \leq \mu(X \setminus K) < \epsilon$ for all $P \in \bigvee_{i=1}^{n-1} T^i(\mathcal{P})$. Also note that, since each $P \in \bigvee_{i=1}^{n-1} T^i(\mathcal{P})$ intersects at most one of $H_1, \ldots, H_m$, the partition $\{P \cap K : P \in \bigvee_{i=1}^{n-1} T^i(\mathcal{P})\}$ of K refines $\{H_i : 1 \leq i \leq m\}$, and so we infer

$$\begin{aligned}
-\sum_{i=1}^{m} \mu(H_i) \log \mu(H_i) &\leq -\sum_{P \in \bigvee_{i=1}^{n-1} T^i(\mathcal{P})} \mu(P \cap K) \log \mu(P \cap K) \\
&\leq -\sum_{P \in \bigvee_{i=1}^{n-1} T^i(\mathcal{P})} \mu(P) \log \mu(P) + 1.
\end{aligned}$$

Combining the above two estimates we obtain $S(\omega \circ \psi_n) \leq H_\mu(\bigvee_{j=0}^{n-1} T^j(\mathcal{P}), \theta) + 2$, so that

$$cpe(\iota, \omega, \Omega^{(n)}, \delta) \leq H_\mu\Big(\bigvee_{j=0}^{n-1} T^j(\mathcal{P}), \theta\Big) + 2.$$

Dividing by n and taking the lim sup yields $hm_\sigma(\theta, \iota, \omega, \Omega, \delta) \leq H_\mu(\mathcal{P}, \theta)$. Taking the supremum over all $\delta > 0$, $\Omega \in Pf(C(X))$, and $\omega \in \mathfrak{E}(\sigma, \iota)$, we conclude that $hm_\sigma(\theta) = hm_\sigma(\theta, \iota) \leq h_\mu(\theta)$. $\qquad \square$

Proposition 4.13. *(Concavity)* *If $\sum_{i=1}^{k} \lambda_i \sigma_i$ is a convex combination of θ–invariant states σ_i on $\mathcal{A}$ then*

$$\sum \lambda_i hm_{\sigma_i}(\theta) \leq hm_{\sum \lambda_i \sigma_i}(\theta).$$

Proof. Let $\iota : \mathcal{A} \to \mathcal{D}$ be an embedding into an injective C^*–algebra. Set $\sigma = \sum_{i=1}^{k} \lambda_i \sigma_i$. For each $i = 1, \ldots, k$ let $\omega_i \in \mathfrak{E}(\sigma_i, \iota)$. Then the state ω defined by $\sum \lambda_i \omega_i$ lies in $\mathfrak{E}(\sigma, \iota)$, and

$$\frac{1}{n} \sum \lambda_i \log cpe\left(\iota, \omega_i, \Omega^{(n)}, \delta\right) \leq \frac{1}{n} \log cpe\left(\iota, \sum \lambda_i \omega_i, \Omega^{(n)}, \delta\right)$$

by the concavity of $S(\cdot)$ on state spaces of finite-dimensional C^*–algebras. Therefore

$$\sum \lambda_i hm_{\sigma_i}(\theta, \iota, \omega_i, \Omega, \delta) \leq hm_{\sum \lambda_i \sigma_i}\left(\theta, \iota, \sum \lambda_i \omega_i, \Omega, \delta\right)$$

and hence $\sum \lambda_i hm_{\sigma_i}(\theta, \iota, \omega_i) \leq hm_{\sum \lambda_i \sigma_i}(\theta, \iota, \sum \lambda_i \omega_i)$.

Taking the supremum successively for each $i = 1, \ldots, k$ over $\omega_i \in \mathfrak{E}(\sigma_i, \iota)$ yields

$$\sum \lambda_i hm_{\sigma_i}(\theta, \iota) \leq hm_{\sum \lambda_i \sigma_i}(\theta, \iota),$$

establishing the proposition. $\qquad \square$

The following proposition establishes a variational inequality bounding the free energy in a given state by the approximation pressure.

Proposition 4.14. *If $a \in \mathcal{A}_{sa}$ and σ is a θ–invariant state on $\mathcal{A}$, then*

$$P_\theta(a) \geq hm_\sigma(\theta) + \sigma(a).$$

Proof. Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a faithful representation of $\mathcal{A}$. Then $P_\theta(a) = P_\theta(a, \pi)$ and $hm_\sigma(\theta) = hm_\sigma(\theta, \pi)$.

Suppose $\omega \in \mathfrak{E}(\sigma, \pi)$. Let Ω be a set in $Pf(\mathcal{A})$ containing a, and suppose $\delta > 0$ and $n \in \mathbb{N}$. If $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\pi, \Omega^{(n)}, \delta)$ then

$$\begin{aligned}
\log \mathrm{Tr}\, e^{\phi(a^{(n)})} &\geq S(\omega \circ \psi) + (\omega \circ \psi)\big(\phi\big(a^{(n)}\big)\big) \\
&\geq S(\omega \circ \psi) + n(\omega \circ \pi)(a) - n\delta \\
&= S(\omega \circ \psi) + n\sigma(a) - n\delta,
\end{aligned}$$

so that

$$\frac{1}{n} \log Z_{\theta,n}(a, \pi, \Omega, \delta) \geq \frac{1}{n} cpe(\pi, \omega, \Omega, \delta) + \sigma(a) - \delta.$$

Hence $P_\theta(a, \pi, \Omega, \delta) \geq hm_\sigma(\theta, \pi, \omega, \Omega, \delta) + \sigma(a) - \delta$ and therefore $P_\theta(a, \pi) \geq hm_\sigma(\theta, \pi, \omega) + \sigma(a)$.

Taking the supremum over $\omega \in \mathfrak{E}(\sigma, \pi)$ yields $P_\theta(a, \pi) \geq hm_\sigma(\theta, \pi) + \sigma(a)$, thus establishing the proposition. $\qquad \square$

In view of Prop. 4.14 and the fact that the Sauvageot–Thouvenot entropy majorizes the CNT entropy [26], we immediately obtain the following corollary.

Corollary 4.15. *If $\theta \in Aut(\mathcal{A})$ and σ is a θ–invariant state on $\mathcal{A}$, then the variational inequality of the previous proposition also holds when $hm_\sigma(\theta)$ is replaced by the Sauvageot–Thouvenot or CNT entropy.*

Cor. 4.15 leads us to introduce the notion of equilibrium state.

Definition 4.16. *Let a be a self-adjoint element of $\mathcal{A}$, and θ a u.c.p. map of $\mathcal{A}$. An equilibrium state for $(\mathcal{A}, \theta, a)$ is a θ–invariant state σ such that $h_\sigma(\theta) + \sigma(a) = P_\theta(a)$.*

To round out this section we discuss some properties of the Sauvageot–Thouvenot entropy which we will need in Sect. 6. The following has been noted by Neshveyev and Størmer (cf. Lemma 3.5 in [18]).

Proposition 4.17. [18] *Let $\mathcal{A}$ be a unital C^*–algebra endowed with a u.c.p. map θ, and let $\mathcal{B} \subset \mathcal{A}$ be a θ–invariant unital C^*–subalgebra. Then, given $\epsilon > 0$, any invariant state σ on $\mathcal{B}$ extends to an invariant state $\tilde{\sigma}$ on $\mathcal{A}$ in such a way that*

$$h_{\tilde{\sigma}}(\theta) > h_\sigma(\theta \restriction \mathcal{B}) - \epsilon.$$

To establish concavity we need the Donald identity (cf. Prop. 5.23(v) in [19]), namely, if η is a state and $\sigma = \sum_i \alpha_i \sigma_i$ is a convex combination of states then

$$\sum_i \alpha_i S(\eta, \sigma_i) = S(\eta, \sigma) + \sum_i \alpha_i S(\sigma, \sigma_i).$$

Proposition 4.18. *If $\mathcal{A}$ is a unital C^*–algebra, $\theta : \mathcal{A} \to \mathcal{A}$ is a u.c.p. map, and $\alpha\sigma_1 + \beta\sigma_2$ is a convex combination of θ–invariant states on $\mathcal{A}$, then*

$$h_{\alpha\sigma_1+\beta\sigma_2}(\theta) \geq \alpha h_{\sigma_1}(\theta) + \beta h_{\sigma_2}(\theta) - (\alpha\log\alpha + \beta\log\beta).$$

Proof. Set $\sigma = \alpha\sigma_1 + \beta\sigma_2$ and, for $i = 1, 2$, let $(\lambda_i, \mathcal{P}_i)$ be a stationary coupling of $(\mathcal{A}, \theta, \sigma_i)$ with $(\mathcal{C}_i, \varsigma_i, \mu_i)$ which is optimal within ϵ with respect to $h'(\cdot, \cdot)$. We construct a stationary pairing λ of $(\mathcal{A}, \theta, \sigma)$ with a commutative system $(\mathcal{C}, \varsigma, \mu)$ as follows. Set $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$, $\varsigma = \varsigma_1 \oplus \varsigma_2$, $\mu = \alpha\mu_1 + \beta\mu_2$, $\lambda = \alpha\lambda_1 + \beta\lambda_2$, $\mathcal{P} = \mathcal{P}_1 \oplus \mathcal{P}_2$. Then the entropy of this model is

$$\alpha\Big(H_\mu\Big(\mathcal{P}_1 \mid \bigvee_{k=0}^\infty \varsigma^k(\mathcal{P}_1)\Big) - H_\mu(\mathcal{P}_1)\Big) + \beta\Big(H_\mu\Big(\mathcal{P}_2 \mid \bigvee_{k=0}^\infty \varsigma^k(\mathcal{P}_2)\Big) - H_\mu(\mathcal{P}_2)\Big)$$

$$-\alpha\log\alpha - \beta\log\beta + \alpha \sum_{p\in\mathcal{P}_1} \mu_1(p)S(\sigma, \sigma_p) + \beta \sum_{p\in\mathcal{P}_2} \mu_2(p)S(\sigma, \sigma_p).$$

By the Donald identity, if η is any state on $\mathcal{A}$,

$$\alpha \sum_{p\in\mathcal{P}_1} \mu_1(p)S(\sigma, \sigma_p) + \beta \sum_{p\in\mathcal{P}_2} \mu_2(p)S(\sigma, \sigma_p)$$

$$= -S(\eta, \sigma) + \alpha \sum_{p\in\mathcal{P}_1} \mu_1(p)S(\eta, (\sigma_1)_p)$$

$$+ \beta \sum_{p\in\mathcal{P}_2} \mu_2(p)S(\eta, (\sigma_2)_p)$$

$$\geq -\alpha S(\eta, \sigma_1) - \beta S(\eta, \sigma_2) + \alpha \sum_{p\in\mathcal{P}_1} \mu_1(p)S(\eta, (\sigma_1)_p)$$

$$+ \beta \sum_{p\in\mathcal{P}_2} \mu_2(p)S(\eta, (\sigma_2)_p)$$

by the joint convexity of the relative entropy. Again applying the Donald identity, the last term coincides with

$$\alpha \sum_{p \in \mathcal{P}_1} \mu_1(p) S(\sigma_1, (\sigma_1)_p) + \beta \sum_{p \in \mathcal{P}_2} \mu_2(p) S(\sigma_2, (\sigma_2)_p).$$

$\square$

Lemma 4.19. *Let $\theta : \mathcal{A} \to \mathcal{A}$ be a u.c.p. map, and assume that σ is θ^r–invariant for some $r \in \mathbb{N}$. Then, for all $j = 1, \ldots, r - 1$,*

$$h_{\sigma \circ \theta^j}(\theta^r) \geq h_\sigma(\theta^r).$$

Proof. If λ is an optimal stationary pairing of $(\mathcal{A}, \sigma, \theta^r)$ with $(\mathcal{C}, \mu, \varsigma)$ and $\mathcal{P}$ is a finite partition of projections in $\mathcal{C}$ then $\lambda_j := (\lambda \circ \theta^j) \otimes \varsigma$ defines a stationary pairing of $(\mathcal{A}, \sigma \theta^j, \theta^r)$ with $(\mathcal{C}, \mu, \varsigma)$. The monotonicity of the quantum relative entropy yields

$$S(\sigma \circ \theta^j, \sigma_p \circ \theta^j) \geq S(\sigma \circ \theta^j \circ \theta^{r-j}, \sigma_p \circ \theta^j \circ \theta^{r-j}) = S(\sigma, \sigma_p),$$

establishing the inequality. $\square$

5   Cuntz–Krieger and Crossed Product Algebras

In this section we examine pressure in Cuntz–Krieger and crossed product algebras, exercising in different directions their respective structures as Pimsner algebras (see [20] and Sect. 6).

1. Cuntz–Krieger algebras

In classical ergodic theory the variational principle was first proved for lattice systems by Ruelle [22, 24] (see also [25]). If we assume for simplicity that the system is one-dimensional, then it is isomorphic to a subshift of finite type [25]. The partition function corresponding to the classical pressure then takes the simple form

$$Z_n(f) = \sum_C e^{\max\{\sum_{j=0}^{n-1} f \circ T^j(x) : x \in C\}}$$

where the sum is taken over all cylinders C of the subshift obtained by fixing the first n coordinates. Inspired by this, we consider the Cuntz–Krieger algebra $\mathcal{O}_A$ (a "noncommutative subshift of finite type") associated to a matrix $A \in M_d(\{0, 1\})$ with no row or column identically zero, introduced by Cuntz and Krieger [10]. $\mathcal{O}_A$ is generated by partial isometries $s_1, \ldots, s_d$ satisfying

$$\sum_i s_i s_i^* = I$$

$$s_i^* s_i = \sum_j A_{ij} s_j s_j^*.$$

Let Λ_A be the one-sided Markov subshift defined by A:

$$\Lambda_A := \{(x_i)_i \in \{1, \ldots, d\}^{\mathbb{N}} : A_{x_i x_{i+1}} = 1, i \in \mathbb{N}\}.$$

The commutative algebra $\mathcal{C}(\Lambda_A)$ of complex-valued continuous functions on Λ_A sits naturally inside $\mathcal{O}_A$ as the C^*–subalgebra generated by the range projections of the iterated products $s_{i_1} \cdots s_{i_r}$. The shift epimorphism $T : (x_1, x_2, \ldots) \in \Lambda_A \to (x_2, x_3, \ldots) \in \Lambda_A$ corresponds to the restriction to $\mathcal{C}(\Lambda_A)$ of the u.c.p. map

$$\theta : t \in \mathcal{O}_A \to \sum s_i t s_i^* \in \mathcal{O}_A.$$

(It suffices to check this on the set $\{s_{i_i} \cdots s_{i_r}(s_{i_1} \cdots s_{i_r})^*\}$, which is total in $\mathcal{C}(\Lambda_A)$). Let $\alpha = (i_1, \ldots, i_r)$ be a finite word of lenth $r =: |\alpha|$ occurring in some element of Λ_A; then $s_\alpha := s_{i_1} \cdots s_{i_r} \neq 0$. Let $[\alpha]$ denote the cylinder set of Λ_A given by

$$[\alpha] := \{(x_i)_i \in \Lambda_A : x_1 = i_1, \ldots, x_r = i_r\}.$$

Lemma 5.1. *If f is a continuous function on Λ_A, then*

$$s_\alpha^* f s_\beta = 0, \quad |\alpha| = |\beta|, \ \alpha \neq \beta,$$
$$s_\alpha^* f s_\alpha(x) = A_{i_r x_1} f(\alpha x),$$

and thus, for $f \geq 0$,

$$s_\alpha^* f s_\alpha \leq \max\{f(x) : \ x \in [\alpha]\} I.$$

These computations turn out to be useful in proving the following result.

Theorem 5.2. *For any self-adjoint element f of $\mathcal{O}_A$ belonging to the subalgebra $\mathcal{C}(\Lambda_A)$, the pressure of f with respect to θ equals the classical pressure of f with respect to the shift T, i.e.,*

$$P_\theta(f) = p_T(f).$$

Proof. By the additivity of both P_θ and p_T under the addition of scalars (Prop. 3.1(b)), we can assume $f \geq 0$. Furthermore, by monotonicity (Prop. 3.3) and Prop. 3.6,

$$p_T(f) = P_{\theta \upharpoonright \mathcal{C}(\Lambda_A)} \leq P_\theta(f).$$

We are thus left to show that $P_\theta(f) \leq p_T(f)$. By Prop. 2.3, we can use the non-unital exact definition of pressure. We generalize the arguments by Boca and Goldstein [4] for the Voiculescu–Brown entropy. Consider a finite set of the form $\Omega(n_0) = \{s_\alpha p_i s_\beta^*, |\beta| \leq |\alpha| \leq n_0\}$ where $p_i := s_i s_i^*$ and α and β are finite words appearing in Λ_A. Consider the contractive c.p. maps $\rho_m : \mathcal{O}_A \to M_{\vartheta_m}(\mathcal{O}_A)$,

$t \to (s_\mu^* t s_\nu)$, where ϑ_m is the number of blocks occurring in elements of Λ_A of length m. By Lemma 2 in [4], for $m \geq n + n_0$, $j = 0, \ldots, n-1$, and $t = s_\alpha p_i s_\beta^* \in \Omega(n_0)$, if $|\beta| < |\alpha|$ then

$$\rho_m(\theta^j(t)) = \sum_{|\mu|=|\alpha|-|\beta|} x(\mu) \otimes s_\mu$$

while if $|\alpha| = |\beta|$ then

$$\rho_m(\theta^j(t)) = \sum_{j=1}^d x(j) \otimes s_j^* s_j,$$

where $x(\mu)$ and $x(j)$ are partial isometries of $M_{\vartheta_m}(\mathbb{C})$ depending also on i, α, and β. Given $\delta > 0$ consider $(\phi_0, \psi_0, M_{m_0}(\mathbb{C}))$ such that

$$\|(\psi_0 \circ \phi_0)(s_r^* s_r) - s_r^* s_r\| + \|(\psi_0 \circ \phi_0)(s_\gamma) - s_\gamma\| < \frac{\delta}{\max(d, \vartheta_{n_0})}$$

for $|\gamma| \leq n_0$ and $r = 1, \ldots, d$. Such a triple exists because $\mathcal{O}_A$ is nuclear. Then by the proof of Prop. 3 in [4] one can produce an element $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}_0(\pi, \Omega(n_0)^{(n)}, \delta)$ by setting $\phi := (\iota \otimes \phi_0) \circ \rho_{n+n_0}$ and $\psi := \psi_{n+n_0} \circ (\iota \otimes \psi_0)$. Here $\psi_m : M_{\vartheta_m}(\mathcal{O}_A) \to \mathcal{O}_A$ takes the matrix $(t_{\alpha\beta})$ to $\sum s_\alpha t_{\alpha\beta} s_\beta^*$. We thus compute, by virtue of the previous lemma,

$$\mathrm{Tr}\, e^{\phi(f^{(n)})} = \sum_{\substack{\alpha \text{ a word in } \Lambda_A \text{ of} \\ \text{length } n + n_0}} \mathrm{Tr}\, e^{\phi_0(s_\alpha^* f^{(n)} s_\alpha)}$$

$$\leq m_0 \sum_{|\alpha|=n+n_0} e^{\max\{f^{(n)}(x), x \in [\alpha]\}}$$

$$\leq m_0 d^{n_0} \sum_{|\alpha|=n} e^{\max\{f^{(n)}(x), x \in [\alpha]\}},$$

and therefore by the computation of pressure for (finite type) subshifts (see, e.g., [11]) we obtain

$$P_\theta(f, \Omega(n_0), \delta) = P_\theta^0(f, \Omega(n_0), \delta) \leq p_T(f).$$

This inequality implies by the Kolmogorov–Sinai property (Prop. 3.4) that $P_\theta(f) \leq p_T(f)$, thus completing the proof. $\qquad\square$

Remark. It is not surprising to note that the above theorem produces as a special case Boca and Goldstein's result: $ht(\theta) = h_{\mathrm{top}}(\Lambda_A) = \log r(A)$ [4].

We conclude this subsection with a discussion of the variational principle in Cuntz–Krieger algebras, comparing $P_\theta(f)$ with the free energies $h_\sigma(\theta) + \sigma(f)$, where $h_\sigma(\theta)$ denotes the CNT entropy of θ computed with respect to a θ–invariant state σ.

We shall need the following lemma, proven, in a more general form, in [21].

Lemma 5.3. *Any θ–invariant state σ on $\mathcal{O}_A$ containing $\mathcal{C}(\Lambda_A)$ in its centralizer satisfies*

$$h_\mu(T) \leq h_\sigma(\theta)$$

where μ is the T–invariant measure on Λ_A obtained restricting σ. Furthermore any faithful T–invariant measure μ arises in this way.

We have thus obtained the following result.

Theorem 5.4. *Let f be as in Theorem 5.2. Let σ be a θ–invariant state of $\mathcal{O}_A$ centralized by $\mathcal{C}(\Lambda_A)$, and μ the shift–invariant measure on Λ_A obtained restricting σ to $\mathcal{C}(\Lambda_A)$. Then*

$$h_\mu(T) + \mu(f) \leq h_\sigma(\theta) + \sigma(f) \leq P_\theta(f) = p_T(f).$$

Therefore, by Lemma 5.3, if (Λ_A, T, f) admits a faithful equilibrium measure μ, such a measure extends to an equilibrium state σ for $(\mathcal{O}_A, \theta, f)$.

Proof. Combine the previous Lemma with Theorem 5.2. $\qquad\qquad\square$

2. Crossed Products

Now we turn to crossed products and establish a generalization to pressure of a result of Brown [7] which asserts that the Voiculescu–Brown entropy of an automorphism of a unital exact C^*–algebra remains constant under passing to the induced inner automorphism on the crossed product. Our proof follows Brown's approach, which in turn is based on a construction of Sinclair and Smith [27].

Theorem 5.5. *If $\mathcal{A}$ is a unital exact C^*-algebra, $\theta \in Aut(\mathcal{A})$, a is a self-adjoint element in $\mathcal{A}$, and u is the canonical unitary implementing θ in $\mathcal{A} \rtimes_\theta \mathbb{Z}$, then*

$$P_\theta(a) = P_{\mathrm{Ad}\,u}(a),$$

where a has been identified on the right with its image under the natural inclusion $\mathcal{A} \hookrightarrow \mathcal{A} \rtimes_\theta \mathbb{Z}$.

Proof. Without loss of generally we may identify $\mathcal{A}$ with its image under a faithful unital representation on a Hilbert space $\mathcal{H}$ and, letting $\pi : \mathcal{A} \to \mathcal{B}(l^2(\mathbb{Z}, \mathcal{H}))$ be the $*$-monomorphism defined as on p. 16 of [7], identify $\mathcal{A} \rtimes_\theta \mathbb{Z}$ with the C^*-algebra generated by $\pi(\mathcal{A})$ and the image of the amplified left regular representation λ of $\mathbb{Z}$ in $\mathcal{B}(l^2(\mathbb{Z}, \mathcal{H}))$.

The inequality $P_\theta(a) \leq P_{\mathrm{Ad}\,u}(\pi(a))$ is an immediate consequence of monotonicity (Prop. 3.3).

To establish the reverse inequality, we adapt the proof of Brown [7] for entropy. Let $\Omega \in Pf(\mathcal{A} \rtimes_\theta \mathbb{Z})$ be of the form $\{\pi(x_1)\lambda_{n_1}, \ldots, \pi(x_l)\lambda_{n_l}\}$ with $\|x_j\| \leq 1$ for $j = 1, \ldots, l$, as in the proof of Theorem 3.5 in [7]. Note that the span of such sets is dense in $\mathcal{A} \rtimes_\theta \mathbb{Z}$, and so by Prop. 3.4 we need only show that $P_{\mathrm{Ad}\,u}(id_{\mathcal{A} \rtimes_\theta \mathbb{Z}}, a, \Omega) \leq P_\theta(a)$. By Lemma 3.4 in [7] there exist a finite set $F \in \mathbb{Z}$ and $\Omega' \in Pf(\mathcal{A})$ such that, if $n \geq 0$, then

$$(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(id_\mathcal{A}, (\Omega')^{(n)}, \delta)$$

implies

$$(\phi', \psi', \mathcal{F} \otimes \mathcal{B}) \in \mathrm{CPA}(id_{\mathcal{A} \rtimes_\theta \mathbb{Z}}, \Omega^{(n)}, 2\delta)$$

where ϕ' is the u.c.p. map $x \mapsto (1 \otimes \phi)((p_F \otimes 1)(x)(p_F \otimes 1))$, ψ' is a u.c.p. map, and $\mathcal{F}$ is the finite-dimensional C^*-algebra $p_F \mathcal{B}(l^2(\mathbb{Z}))p_F$, with p_F denoting the projection from $l^2(\mathbb{Z})$ onto $span\{\xi_t :$

$t \in F\}$. We can assume that F is of the form $\{-m, -m+1, \ldots, -1, 0, 1, \ldots, m-1, m\}$ for some positive integer m. By Lemma 3.1 in [7], $\pi(a) = \sum_{t \in \mathbb{Z}} e_{t,t} \otimes \theta^{-t}(a)$, where the convergence is in the strong operator topology.

Suppose now that $n \geq 2m+1$ and $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(id_{\mathcal{A}}, (\Omega')^{(n)}, \delta)$. Since $\phi'(\sum_{i=0}^{n-1} \mathrm{Ad}\, u^i(\pi(a))) = \sum_{i=0}^{n-1} \sum_{t=-m}^{m} p_F e_{t,t} p_F \otimes (\phi \circ \theta^{-t-i})(a)$ we have

$$\left\| \phi' \left(\sum_{i=0}^{n-1} \mathrm{Ad}\, u^i(\pi(a)) \right) - \sum_{k=m}^{n-m-1} \sum_{t=-m}^{m} p_F e_{t,t} p_F \otimes \left(\phi \circ \theta^{-k} \right)(a) \right\|$$

$$= \left\| \sum_{k=-m}^{m-1} \sum_{t=-m}^{k} p_F e_{t,t} p_F \otimes \left(\phi \circ \theta^{-k} \right)(a) + \right.$$

$$\left. \sum_{k=n-m}^{n+m-1} \sum_{t=k-n+1}^{m} p_F e_{t,t} p_F \otimes \left(\phi \circ \theta^{-k} \right)(a) \right\|$$

$$\leq \sum_{k=-m}^{m-1} \sum_{t=-m}^{k} \left\| p_F e_{t,t} p_F \right\| \left\| \left(\phi \circ \theta^{-k} \right)(a) \right\| +$$

$$\sum_{k=n-m}^{n+m-1} \sum_{t=k-n+1}^{m} \left\| p_F e_{t,t} p_F \right\| \left\| \left(\phi \circ \theta^{-k} \right)(a) \right\|$$

$$\leq 4m^2 \|a\|$$

and so

$$\left| \log \mathrm{Tr}_{\mathcal{F} \otimes \mathcal{B}} e^{\phi'\left(\sum_{i=0}^{n-1} \mathrm{Ad}\, u^i(\pi(a)) \right)} \right.$$

$$\left. - \log \mathrm{Tr}_{\mathcal{F} \otimes \mathcal{B}} e^{\sum_{k=m}^{n-m-1} \sum_{t=-m}^{m} p_F e_{t,t} p_F \otimes \left(\phi \circ \theta^{-k} \right)(a)} \right| \leq 4m^2 \|a\|$$

by Cor. 3.15 in [19]. Next observe that if β and γ are maximal sets of pairwise orthogonal spectral projections for $\sum_{t=-m}^{m} p_F e_{t,t} p_F$ and $\sum_{k=m}^{n-m-1} (\phi \circ \theta^{-k})(a)$, respectively, then $\beta \otimes \gamma$ is a maximal set of pairwise orthogonal spectral projections for $\sum_{k=m}^{n-m-1} \sum_{t=-m}^{m} p_F e_{t,t} p_F \otimes (\phi \circ \theta^k)(a)$, and thus

$$\mathrm{Tr}_{\mathcal{F} \otimes \mathcal{B}} \, e^{\sum_{k=m}^{n-1-m} \sum_{t=-m}^{m} p_F e_{t,t} p_F \otimes \left(\phi \circ \theta^{-k} \right)(a)}$$

$$= \sum_{e \in \beta} \sum_{f \in \gamma} e^{\mathrm{Tr}_{\mathcal{F} \otimes \mathcal{B}} \left[(e \otimes f) \left(\sum_{k=m}^{n-m-1} \sum_{t=-m}^{m} p_F e_{t,t} p_F \otimes \left(\phi \circ \theta^{-k} \right)(a) \right) \right]}$$

$$= \sum_{e \in \beta} \sum_{f \in \gamma} e^{\mathrm{Tr}_{\mathcal{F} \otimes \mathcal{B}} \left[(e \otimes f) \left(\left(\sum_{t=-m}^{m} p_F e_{t,t} p_F \right) \otimes \left(\sum_{k=m}^{n-m-1} \left(\phi \circ \theta^{-k} \right)(a) \right) \right) \right]}$$

$$= \sum_{e \in \beta} \sum_{f \in \gamma} e^{\mathrm{Tr}_{\mathcal{F} \otimes \mathcal{B}} \left[(e \otimes f) \left(1_{\mathcal{F}} \otimes \sum_{k=m}^{n-m-1} \left(\phi \circ \theta^{-k} \right)(a) \right) \right]}$$

$$= \sum_{e \in \beta} \sum_{f \in \gamma} e^{\mathrm{Tr}_{\mathcal{F}} \left[e \cdot 1_{\mathcal{F}} \right] \mathrm{Tr}_{\mathcal{B}} \left[f \left(\sum_{k=m}^{n-m-1} \left(\phi \circ \theta^{-k} \right)(a) \right) \right]}$$

$$= card(\beta) \sum_{f \in \gamma} e^{\mathrm{Tr}_{\mathcal{B}} \left[f \left(\sum_{k=m}^{n-m-1} \left(\phi \circ \theta^{-k} \right)(a) \right) \right]}$$

$$= (2m+1) \sum_{f \in \gamma} e^{\mathrm{Tr}_{\mathcal{B}} \left[f \left(\sum_{k=m}^{n-m-1} \left(\phi \circ \theta^{-k} \right)(a) \right) \right]}$$

$$= (2m+1) \, \mathrm{Tr}_{\mathcal{B}} \, e^{\sum_{k=m}^{n-m-1} \left(\phi \circ \theta^{-k} \right)(a)}.$$

Furthermore, another application of Cor. 3.15 in [19] yields

$$\left| \log \operatorname{Tr}_{\mathcal{B}} e^{\phi\left(\sum_{k=0}^{n-1} \theta^{-k}(a)\right)} - \log \operatorname{Tr}_{\mathcal{B}} e^{\phi\left(\sum_{k=m}^{n-m-1} \theta^{-k}(a)\right)} \right|$$

$$\leq \left\| \phi \left(\sum_{k=0}^{n-1} \theta^{-k}(a) - \sum_{k=0}^{m-1} \theta^{-k}(a) \right) \right\|$$

$$\leq \sum_{k=0}^{m-1} \left\| \left(\phi \circ \theta^{-k} \right)(a) \right\| + \sum_{k=n-m}^{n-1} \left\| \left(\phi \circ \theta^{-k} \right)(a) \right\|$$

$$\leq 2m.$$

Combining these estimates we obtain

$$\left| \log \operatorname{Tr}_{\mathcal{F}\otimes\mathcal{B}} e^{\phi'\left(\sum_{i=0}^{n-1} \operatorname{Ad} u^i(\pi(a))\right)} - \log \operatorname{Tr}_{\mathcal{B}} e^{\phi\left(\sum_{k=0}^{n-1} \theta^{-k}(a)\right)} \right|$$

$$\leq \left| \log \operatorname{Tr}_{\mathcal{F}\otimes\mathcal{B}} e^{\phi'\left(\sum_{i=0}^{n-1} \operatorname{Ad} u^i(\pi(a))\right)} \right.$$

$$\left. - \log \operatorname{Tr}_{\mathcal{F}\otimes\mathcal{B}} e^{\sum_{k=m}^{n-m-1} \sum_{t=-m}^{m} p_F e_{t,t} p_F \otimes \left(\phi\circ\theta^{-k}\right)(a)} \right| +$$

$$\left| \log \operatorname{Tr}_{\mathcal{F}\otimes\mathcal{B}} e^{\sum_{k=m}^{n-m-1} \sum_{t=-m}^{m} p_F e_{t,t} p_F \otimes \left(\phi\circ\theta^{-k}\right)(a)} \right.$$

$$\left. - \log \operatorname{Tr}_{\mathcal{B}} e^{\sum_{k=m}^{n-m-1} \left(\phi\circ\theta^{-k}\right)(a)} \right| +$$

$$\left| \log \operatorname{Tr}_{\mathcal{B}} e^{\sum_{k=m}^{n-m-1} \left(\phi\circ\theta^{-k}\right)(a)} - \log \operatorname{Tr}_{\mathcal{B}} e^{\phi\left(\sum_{k=0}^{n-1} \theta^{-k}(a)\right)} \right|$$

$$\leq 2m(2m+1)\|a\| + \log(2m+1).$$

Since the above holds for any $(\phi, \psi, B) \in \mathrm{CPA}(id_{\mathcal{A}}, \mathcal{A}, (\Omega')^{(n)}, \delta)$, it follows that

$$\log Z_{\operatorname{Ad} u, n}(id_{\mathcal{A} \rtimes_\theta \mathbb{Z}}, \pi(a), \Omega, 2\delta)$$
$$\leq \log Z_{\theta, n}(id_{\mathcal{A}}, a, \Omega', \delta) + 2m(2m+1)\|a\| + \log(2m+1).$$

Thus, letting n vary while m remains fixed, we infer that

$$P_{\operatorname{Ad} u}(id_{\mathcal{A} \rtimes_\theta \mathbb{Z}}, \pi(a), \Omega, 2\delta) \leq P_\theta(id_{\mathcal{A}}, a, \Omega', \delta)$$

We conclude that $P_{\operatorname{Ad} u}(id_{\mathcal{A} \rtimes_\theta \mathbb{Z}}), \pi(a), \Omega) \leq P_\theta(id_{\mathcal{A}}, a)$, completing the proof of the theorem. $\square$

6   Variational Principle for bimodule algebras

This section is divided into three subsections. In the first subsection we obtain a variational principle for a class of C^*–algebras generated by a Hilbert bimodule [20] which generalizes the results of subsection 5.1, in the second subsection we discuss equilibrium states for the same class, and in the last subsection we discuss an application to Matsumoto C^*–algebras associated to a subshift.

Let $\mathcal{A}$ be a unital exact C^*–algebra faithfully represented on a Hilbert space, and X a Hilbert $\mathcal{A}$–bimodule, i.e. X is a right Hilbert $\mathcal{A}$–module endowed with a faithful left action of $\mathcal{A}$ given by

a unital *–monomorphism $\mathcal{A} \to \mathcal{L}_{\mathcal{A}}(X)$ into the algebra of right $\mathcal{A}$–linear endomorphismsms of $\mathcal{A}$. We will always assume that X is finitely generated as a right module. Following [20], we construct the universal C^*–algebra algebra $\mathcal{O}_X$ generated by X and a unital copy of $\mathcal{A}$ satisfying

$$x^* a x' = <x, ax'>, \quad x, x' \in X, a \in \mathcal{A},$$
$$\sum_i x_i x_i^* = I, \quad \{x_i\}_i \text{ a basis of } X,$$

where a basis of X is a finite subset such that

$$x = \sum_i x_i <x_i, x>, \quad x \in X.$$

The Banach subspaces $\mathcal{L}_{\mathcal{A}}(X^{\otimes r}, X^{\otimes s})$ are isometrically embedded in $\mathcal{O}_X$ in a manner respecting the inclusions $\mathcal{L}_{\mathcal{A}}(X^{\otimes r}, X^{\otimes s}) \hookrightarrow \mathcal{L}_{\mathcal{A}}(X^{\otimes r+1}, X^{\otimes s+1})$ under which $T \in \mathcal{L}_{\mathcal{A}}(X^{\otimes r}, X^{\otimes s})$ is identified with $T \otimes 1_X \in \mathcal{L}_{\mathcal{A}}(X^{\otimes r+1}, X^{\otimes s+1})$. The algebra $\mathcal{O}_X$ carries an automorphic action $\gamma : \mathbb{T} \to \mathcal{O}_X$ of the circle given by

$$\gamma_z(x) = zx, \quad z \in \mathbb{T},$$
$$\gamma_z(a) = a, \quad a \in \mathcal{A}$$

and referred to as the gauge action. The corresponding fixed-point algebra will be denoted by $\mathcal{O}_X^{(0)}$. It is a fact that $\mathcal{O}_X$ is an exact C^*–algebra if $\mathcal{A}$ is exact. This property has been proven in [13] for general Hilbert bimodules. However, for the class of special modules we will be considering, exactness of $\mathcal{O}_X$, under the corresponding assumption for the coefficient algebra, will result from the proof of Prop. 6.9. As noticed in [20], if α is an automorphism of $\mathcal{A}$ and $X = \mathcal{A}$ with right Hilbert $\mathcal{A}$–module structure

$$xa = x\alpha(a),$$
$$<x, y> = \alpha(x^* y), \quad x, y \in X, a \in \mathcal{A}$$

and left action given by left multiplication, then $\mathcal{O}_X = \mathcal{A} \rtimes_\alpha \mathbb{Z}$. If $\mathcal{A}$ is commutative and finite-dimensional, $\mathcal{O}_X$ is a Cuntz–Krieger algebra. Matsumoto algebras associated to subshifts [16] also arise as Pimsner algebras [21]. We further assume that X admits a basis $\{x_i\}_i$ such that

$$<x_i, ax_j> = 0, \quad i \neq j, a \in \mathcal{A}.$$

In other words, as a right $\mathcal{A}$–module, $X = q_1 \mathcal{A} \oplus \cdots \oplus q_d \mathcal{A}$, where each q_i is a projection of $\mathcal{A}$, while the left $\mathcal{A}$–action is defined by the diagonal action of *–monomorphisms $\rho_i : \mathcal{A} \to \mathcal{A}$ with

$$\rho_i(I) = q_i. \tag{6.1}$$

Thus $\mathcal{O}_X$ is the universal C^*–algebra generated by a unital copy of $\mathcal{A}$ and partial isometries $x_1, \ldots, x_d$ satisfying

$$ax_i = x_i \rho_i(a), \quad i = 1, \ldots, d, a \in \mathcal{A}, \tag{6.2}$$

$$\sum x_i x_i^* = I, \tag{6.3}$$

$$x_i^* x_i = q_i. \tag{6.4}$$

All of the bimodules in the above examples admit an orthogonal basis. We recall for convenience Pimsner's construction of the bimodule generating $\mathcal{O}_A$ [20]. Take $\mathcal{A} = \oplus_1^d \mathbb{C}$ as the coefficient algebra, and let $A = (A_{rs}) \in M_d(\{0,1\})$. Then if $\{p_i\}$ stands for the set of minimal projections of $\mathcal{A}$, X is the bimodule $q_1 \mathcal{A} \oplus \cdots \oplus q_d \mathcal{A}$, where $q_i = \sum_j A_{ij} p_j$, and $\rho_i(p_j) = \delta_{i,j} q_i$. The corresponding basis elements $\{x_i\}$, usually denoted by $\{s_i\}$, generate $\mathcal{O}_X$ and satisfy

$$\sum_i s_i s_i^* = I,$$

$$s_i^* s_i = q_i,$$

$$s_i s_i^* = p_i.$$

1. The variational principle

Our first aim is to give an upper and lower bound for the pressure of a self-adjoint element a in a suitable C^*–subalgebra of $\mathcal{O}_X$ with respect to the u.c.p. map

$$\theta : b \to \sum x_i b x_i^*$$

on $\mathcal{O}_X$. These bounds will lead, under certain circumstances, to a computation of $P_\theta(a)$ and to the variational principle.

Remark. In contrast with the class of C^*–algebras considered by Neshveyev and Størmer [18], $(\mathcal{O}_X, \theta)$ is usually not asymptotically Abelian. Indeed, if $q_1 + \cdots + q_d$ is invertible, θ restricts to a unique monomorphism σ of $\mathcal{A}' \cap \mathcal{O}_X$ such that $\sigma(t)x = xt$ for $t \in \mathcal{A}' \cap \mathcal{O}_X$ and $x \in X$ [21]. If $\mathcal{O}_X$ is simple and there is a nonscalar element $t \in \mathcal{A}' \cap \mathcal{O}_X$, one has $x_i \sigma^r(t) = \sigma^{r+1}(t)x_i$, and therefore $[x_i, \theta^r(t)] = (\sigma^{r+1}(t) - \sigma^r(t))x_i$. If this tended to 0 for all i, then one would have $\sigma(t) = t$, so that t would be an element in the centre of $\mathcal{O}_X$.

If $\alpha = (i_1, \ldots, i_r)$ we write x_α for $x_{i_1} \ldots x_{i_r}$ and denote by $|\alpha|$ the length r of α. We will restrict a to be an element of the following amplification of the coefficient algebra:

$$\mathcal{D} := \left\{ b \in \mathcal{O}_X^{(0)} : x_\alpha^* b x_\beta = 0, |\alpha| = |\beta|, \alpha \neq \beta \right\}.$$

Note that $\mathcal{D}$ is a unital C^*–subalgebra containing $\mathcal{A}$ and elements of the form $x_\alpha a x_\alpha^*$, $a \in \mathcal{A}$, and is invariant under θ and the maps $\mathrm{Ad}\, x_i^*$, $i = 1, \ldots, d$. Furthermore $\mathcal{D}$ is an exact C^*–algebra since it is a subalgebra of $\mathcal{O}_X^{(0)}$, which is exact. Notice that the $\mathrm{Ad}\, x_i^*$'s restrict to endomorphisms of $\mathcal{D}$. The closed subspace $X_\mathcal{D} = X\mathcal{D}$ of $\mathcal{O}_X$ is a Hilbert bimodule over $\mathcal{D}$ isomorphic to $q_1 \mathcal{D} \oplus \cdots \oplus q_d \mathcal{D}$ as a right Hilbert module with diagonal left action induced by $\mathrm{Ad}\, x_i^*$, $i = 1, \ldots, d$. One has $\mathcal{O}_{X_\mathcal{D}} = \mathcal{O}_X$. This construction is familiar in the case of Cuntz–Krieger algebras.

Proposition 6.1 *If X is the Hilbert bimodule defining the Cuntz–Krieger algebra $\mathcal{O}_A$, $\mathcal{D}$ is the Abelian C^*–subalgebra $\mathcal{C}(\Lambda_A)$.*

Proof. The inclusion $\mathcal{C}(\Lambda_A) \subset \mathcal{D}$ follows from the fact that $\mathcal{D}$ is θ–invariant and contains the range projections p_i, $i = 1, \ldots, d$. To show the opposite inclusion we consider a sequence of conditional expectations $(E_r)_r$ onto the finite-dimensional C^*–subalgebras $\mathcal{F}_r$ generated by $\{s_\alpha p_i s_\beta^*, |\alpha| = |\beta| = r, i = 1, \ldots, d\}$. We choose each E_r to be invariant under a faithful trace of $\mathcal{O}_A^{(0)}$ obtained restricting a β–KMS state ω of $\mathcal{O}_A$ for the one-parameter group $t \to \gamma_{e^{2\pi it}}$. Then the KMS condition

$$\omega(s_i^* t) = e^\beta \omega(t s_i^*), \quad t \in \mathcal{O}_A,$$

yields

$$s_i^* E_{r+1}(t) s_j = E_r(s_i^* t s_j), \quad t \in \mathcal{O}_A^{(0)}.$$

Thus if $t \in \mathcal{D}$ then $E_r(t) = \sum_{|\alpha|=r} s_\alpha E_0(s_\alpha^* t s_\alpha) s_\alpha^*$, which is contained in $\mathcal{C}(\Lambda_A)$ since the range of E_0 is the linear span of the range projections $p_i = s_i s_i^*$, $i = 1, \ldots, d$. Since $(E_r)_r$ converges to the identity, we have $t \in \mathcal{C}(\Lambda_A)$. $\square$

Note that, on the other hand, if X is a Hilbert bimodule defined by an automorphism α of $\mathcal{A}$, then $\mathcal{D} = \mathcal{A}$.

Recall that a one-sided subshift is a closed subset of the compact space $\{1, \ldots, d\}^\mathbb{N}$ such that $T(\Lambda) = \Lambda$, where $T((a_k)_k) = (a_{k+1})_k$ is the left shift epimorphism of the full shift space. Let $\Lambda^{(n)}$ stand for the set of n-tuples $\alpha = (i_1, \ldots, i_n)$ for which there is $(a_k)_k \in \Lambda$ such that $a_1 = i_1, \ldots, a_n = i_n$, and set $\vartheta_n = \mathrm{Card}\,(\Lambda^{(n)})$.

We associate to an orthogonal basis $\{x_i\}_{i=1}^d$ of a Hilbert bimodule X the set

$$\Lambda_{\{x_i\}} = \{(a_k)_k \in \{1, \ldots, d\}^\mathbb{N} : x_{a_1} \cdots x_{a_n} \neq 0 \text{ for all } n \in \mathbb{N}\}.$$

We will write $\Lambda = \Lambda_{\{x_i\}}$. It easily checked that Λ is a one–sided subshift. The relation $\sum_i x_i x_i^* = I$ shows that $\Lambda \neq \emptyset$. For $\alpha = (i_1, \ldots, i_n) \in \Lambda^{(n)}$ we set $x_\alpha = x_{i_1} \ldots x_{i_n}$, $q_\alpha = x_\alpha^* x_\alpha$, $p_\alpha = x_\alpha x_\alpha *$ and $\rho_\alpha = \rho_{i_1} \ldots \rho_{i_n}$. Note that $\mathcal{C}(\Lambda)$ embeds naturally in $\mathcal{O}_X$ as the $C*$–subalgebra generated by the projections p_α, $\alpha \in \cup \Lambda^{(n)}$.

We define the topological entropy of the the action of X on $\mathcal{A}$ by

$$ht(\mathcal{A}, X) := \sup_{\Omega \in Pf(\mathcal{A})} \sup_{\delta > 0} \limsup_n \frac{1}{n} \log rcp(\mathcal{A}, \Omega^{(n,X)}, \delta),$$

where

$$\Omega^{(n,X)} := \{\rho_\mu(t) : t \in \Omega, |\mu| \leq n - 1\}.$$

Our aim is to prove the following.

Theorem 6.2. *Let $\mathcal{A}$ be a unital exact C^*–algebra and X a Hilbert $\mathcal{A}$–bimodule defined as above by $*$–monomorphisms $\rho_i : \mathcal{A} \to \mathcal{A}$, $i = 1, \ldots, d$, with the property that $\sum_{i=1}^d \rho_i(I)$ is invertible. Consider the C^*–dynamical system $(\mathcal{O}_X, \theta)$. Suppose that $ht(\mathcal{A}, X) = 0$. If $a \in \mathcal{D}$ is positive and satisfies*

$$[a, x_\alpha^* a x_\alpha] = 0$$
$$[a, q_\alpha] = 0$$

for $|\alpha|$ sufficiently large, then

$$P_\theta(a) = \lim_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(n-1)}} e^{\|x_\alpha^* a^{(n)} x_\alpha\|}.$$

(In particular, if $a \in \mathcal{C}(\Lambda)$ then $P_\theta(a)$ coincides with the classical pressure of a w.r.t. the shift on Λ.) Furthermore one has

$$\sup_\sigma h_\sigma(\theta) + \sigma(a) = P_\theta(a),$$

where $h_\sigma(\theta)$ denotes the Sauvageot–Thouvenot entropy of θ and the supremum is taken over all θ–invariant states of $\mathcal{O}_X$.

Before proving the theorem, we discuss an example where the condition $ht(\mathcal{A}, X) = 0$ is easily checked.

Example. Consider the case in which $\mathcal{A}$ is the inductive limit of finite-dimensional C^*–algebras $(\mathcal{F}_r)_{r \in \mathbb{N}}$, each one invariant under $\rho_1, \ldots, \rho_d$, and suppose that $\mathcal{A}$ admits a faithful trace τ. Let Ω be a finite set contained in some $\mathcal{F}_{r_0}$. Then $\Omega^{(n,X)} \subset \mathcal{F}_{r_0}$ for all $n \in \mathbb{N}$. Consider the τ–preserving conditional expectation $E : \mathcal{A} \to \mathcal{F}_{r_0}$ and let $\iota : \mathcal{F}_{r_0} \to \mathcal{A}$ be the inclusion. Clearly $(E, \iota, \mathcal{F}_{r_0}) \in \mathrm{CPA}(\Omega^{(n,X)}, \delta)$ for all $\delta > 0$ and $n \in \mathbb{N}$, and so $ht(\mathcal{A}, X) = 0$.

We shall divide the proof of Theorem 6.2 into three parts. In the first and second part we will give upper and lower bounds for the pressure of a, and in the third part we will prove the variational principle. We start by showing why we require $\sum_i q_i$ to be invertible.

Lemma 6.3. *If $\sum_{i=1}^d q_i$ is invertible, then θ is faithful on $\mathcal{O}_X$.*

Proof. The equality $\theta(t^*t) = 0$ implies $q_i t^* t q_i = x_i^* \theta(t^*t) x_i = 0$, and thus $t q_i = 0$ for all i, so that $t = 0$. $\qquad\qquad\square$

Let $a \in \mathcal{D}$ be a self-adjoint element. We introduce the following notion of pressure for a with respect to the bimodule X. Let $\Omega \in Pf(\mathcal{A})$, $\delta > 0$, and $n \in \mathbb{N}$. Setting $a^{(n)} := \sum_0^{n-1} \theta^j(a)$ as usual, we define the partition function

$$Z_{X,n}(\mathcal{D}, a, \Omega, \delta) =$$
$$\inf \left\{ \sum_{\alpha \in \Lambda^{(n-1)}} \mathrm{Tr}\, e^{\phi\left(x_\alpha^* a^{(n)} x_\alpha\right)} : (\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\mathcal{D}, \Omega^{(n,X)}, \delta) \right\},$$

and the corresponding $P_X(\mathcal{D}, a, \Omega, \delta)$ and $P_X(\mathcal{D}, a, \Omega)$ are obtained in the usual manner. We set

$$P_X(\mathcal{D}, a) = \sup_{\Omega \in Pf(\mathcal{A})} P_X(\mathcal{D}, a, \Omega).$$

We emphasize that this pressure is computed by means of approximations of a faithful representation of $\mathcal{D}$ via factorizations through finite-dimensional C^*–algebras. However, we only let Ω range over finite subsets of $\mathcal{A}$. The definition of $Z_{X,n}(\mathcal{D}, a, \Omega, \delta)$ suggests considering for an element $a \in \mathcal{O}_X$ the sequence

$$\sum_{\alpha \in \Lambda^{(n-1)}} e^{\max \mathrm{spec}\, x_\alpha^* a^{(n)} x_\alpha},$$

which resembles the classical partition function defining the pressure of a subshift (see [11], e.g.).

Lemma 6.4. *If a is a positive element of $\mathcal{O}_X$ then*

$$\lim_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(n-1)}} e^{\|x_\alpha^* a^{(n)} x_\alpha\|} =: \ell$$

exists and

$$h_{\text{top}}(\Lambda) \le \ell \le \|a\| + h_{\text{top}}(\Lambda).$$

Proof. If $|\alpha| = n$ and $|\beta| = m$ then

$$x_{\alpha\beta}^* a^{(n+m+1)} x_{\alpha\beta}$$

$$= x_\beta^* \left(x_\alpha^* a^{(n+1)} x_\alpha \right) x_\beta + x_\beta^* \left(x_\alpha^* \theta^n (\theta(a) + + \cdots + \theta^m(a)) x_\alpha \right) x_\beta$$

$$= x_\beta^* \left(x_\alpha^* a^{(n+1)} x_\alpha \right) x_\beta + x_\beta^* q_\alpha (\theta(a) + \cdots + \theta^m(a)) q_\alpha x_\beta$$

$$= x_\beta^* \left(x_\alpha^* a^{(n+1)} x_\alpha \right) x_\beta + q_{\alpha\beta} \left(x_\beta^* (\theta(a) + \cdots + \theta^m(a)) x_\beta \right) q_{\alpha\beta}$$

since $x_\beta^* q_\alpha = \sum_{|\gamma|=|\beta|} x_\beta^* q_\alpha x_\gamma x_\gamma^* = q_{\alpha\beta} x_\beta^*$. Now the previous term is bounded above by

$$\|x_\alpha^* a^{(n+1)} x_\alpha\| + \|x_\beta^* a^{(m+1)} x_\beta\|$$

and so $s_n := \sum_{\alpha \in \Lambda^{(n)}} e^{\|x_\alpha^* a^{(n+1)} x_\alpha\|}$ satisfies $s_{n+m} \le s_n s_m$. It follows that $\lim \frac{1}{n} \log(s_n)$ exists and equals $\inf \frac{1}{n} \log(s_n)$. The upper and lower bounds for ℓ follow from the inequalities $0 \le \|x_\alpha^* a^{(n)} x_\alpha\| \le n\|a\|$ for $\alpha \in \Lambda^{(n-1)}$. $\square$

Proposition 6.5. *If $a \in \mathcal{D}$ is a self-adjoint element then*

$$P_X(\mathcal{D}, a) \le ht(\mathcal{A}, X) + \lim_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(n-1)}} e^{\max \operatorname{spec} x_\alpha^* a^{(n)} x_\alpha}.$$

Proof. The proof is straightforward once we note that, by Arveson's extension theorem [3], every unital c.p. map $\phi : \mathcal{A} \to M_N(\mathbb{C})$ extends to a unital c.p. map $\tilde{\phi} : \mathcal{D} \to M_N(\mathbb{C})$. $\square$

Before establishing an upper bound for $P_\theta(a)$ for certain $a \in \mathcal{D}$, we shall need a few preliminary results. The first two lemmas are immediate, and so we omit the proofs.

Lemma 6.6. *If $a \in \mathcal{D}$ is a self-adjoint element, and if $\lambda \in \mathbb{R}^+$,*

a) $P_X(\mathcal{D}, a + \lambda) \le P_X(\mathcal{D}, a) + \lambda$,

b) $P_X(\mathcal{D}, a - \lambda) \ge P_X(\mathcal{D}, a) - \lambda$.

Lemma 6.7. *Set*

$$\phi_m : b \in \mathcal{O}_X \to (x_\mu^* b x_\nu)_{\mu,\nu \in \Lambda^{(m)}} \in M_{\vartheta_m}(\mathcal{O}_X).$$

Then for $j = 0, \ldots, n-1$, $|\beta| \leq |\alpha| \leq n_0$, and $t \in \mathcal{A}$ the (μ, ν) entry of

$$\phi_{n+n_0-1}\theta^j(x_\alpha t x_\beta^*)$$

is nonzero only if μ and ν are of the form $\mu = \delta\alpha\mu'$, $\nu = \delta\beta\mu'\nu'$ with $|\delta| = j$ and $|\nu'| = |\alpha| - |\beta|$. The corresponding entry is

$$\rho_{\mu'}(q_{\delta\alpha} t q_{\delta\beta})x_{\nu'}.$$

Lemma 6.8. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a unital C^*–algebra and let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a unital c.p. map. Let x be an element of the unit ball of $\mathcal{A}$ and p, q projections of $\mathcal{A}$ such that $\|\phi(y) - y\| < \delta$ for each $y \in \{x, p, q\}$ and some $\delta < 1$. Then $\|\phi(pxq) - pxq\| < 11\delta^{\frac{1}{2}}$.*

Proof. By Stinespring's theorem [28] there is a Hilbert space $\mathcal{K}$, an isometry $V : \mathcal{H} \to \mathcal{K}$, and a unital *–representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that $\phi(t) = V^*\pi(t)V$, $t \in \mathcal{A}$. In Stinespring's construction $\mathcal{K}$ is the tensor product Hilbert bimodule $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{H}$, where $\mathcal{A}$ is regarded as a $\mathcal{A}$–$\mathbb{C}$ Hilbert bimodule with $\mathcal{A}$–valued inner product defined by $< a, b >= \phi(a^*b)$. One has $\pi(a) = a \otimes I$ for $a \in \mathcal{A}$ and $V\xi = I \otimes \xi$ for $\xi \in \mathcal{H}$. One checks that

$$\begin{aligned}
\|\pi(p)V - Vp\|^2 &\leq \|\phi(p) - \phi(p)p - p\phi(p) + p\| \\
&\leq \|\phi(p) - p\| + \|(p - \phi(p))p\| + \|p(\phi(p) - p)\| \\
&< 3\delta,
\end{aligned}$$

and similarly for q. This implies

$$\begin{aligned}
\|\phi(p)\phi(x)\phi(q) - \phi(pxq)\| &= \|V^*\pi(p)VV^*\pi(x)VV^*\pi(q)V - V^*\pi(pxq)V\| \\
&\leq 4(3\delta)^{\frac{1}{2}} \\
&< 8\delta^{\frac{1}{2}}.
\end{aligned}$$

On the other hand, by our assumption we have

$$\|\phi(p)\phi(x)\phi(q) - pxq\| < 3\delta,$$

which, when combined with the previous estimate, yields the result. $\square$

We are now ready to give an upper bound for $P_\theta(a)$.

Proposition 6.9. *Let X be a Hilbert bimodule over a unital exact C^*–algebra $\mathcal{A}$. Suppose that $X = q_1\mathcal{A} \oplus \cdots \oplus q_d\mathcal{A}$ as a right Hilbert module with left action defined by unital *–monomorphisms $\rho_i : \mathcal{A} \to q_i\mathcal{A}q_i$, $i = 1, \ldots, d$. Then for any self-adjoint element $a \in \mathcal{D}$ commuting asymptotically with the domain projections $\{q_\mu, \mu \in \cup_n\Lambda^{(n)}\}$, i.e.,*

$$\lim_{|\mu| \to \infty} \|[a, \ q_\mu]\| = 0,$$

we have

$$P_\theta(a) \le P_X(\mathcal{D}, a) \le ht(\mathcal{A}, X) + \limsup_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(n-1)}} e^{\max \operatorname{spec} x_\alpha^* a^{(n)} x_\alpha}.$$

Proof. We need only show the first inequality. Let $\Omega \subset \mathcal{A}$ be a finite subset of the unit ball containing I. For $n_0 \in \mathbb{N}$ we set

$$\Omega(n_0) = \{x_\alpha t x_\beta^*, \quad |\beta| \le |\alpha| \le n_0, t \in \Omega\}.$$

Since $\cup_{n_0,\Omega} \Omega(n_0) \cup \Omega(n_0)^*$ is total in $\mathcal{O}_X$, it suffices by monotonicity and the Kolmogorov–Sinai property to show that $P_\theta(a, \Omega(n_0)) \le P_X(\mathcal{D}, a)$ for all $\Omega \in Pf(\mathcal{A})$ and $n_0 \in \mathbb{N}$. Following the proof of Lemma 7.5 in [21], which in turn goes back to [7], given any subset $\Delta \in Pf(\mathcal{O}_X^{(0)})$, $\delta > 0$, and $n_0 \in \mathbb{N}$, we can find a finite subset $F \subset \mathbb{N}$, which depends only on δ and n_0 and not on Δ, such that if

$$(\phi, \psi, \mathcal{B}) \in \operatorname{CPA}\Big(\mathcal{O}_X^{(0)}, \Delta^{(\max F, X)}, \frac{\delta}{2\max_{p \in F} \vartheta_p}\Big)$$

then there is a triple $(\phi', \psi', \mathcal{B}') \in \operatorname{CPA}_0(\mathcal{O}_X, \cup_{|\gamma| \le n_0} \Delta x_\gamma, \delta)$ with $\mathcal{B}' = M_{\vartheta_F} \otimes \mathcal{B}$ and $\phi' = \iota_{M_{\vartheta_F}} \otimes \phi \circ S_F$, where $\vartheta_F = \sum_{p \in F} \vartheta_p$ and

$$S_F : b \in \mathcal{O}_X \to (x_\alpha^* m_{|\alpha|-|\beta|}(b) x_\beta)_{|\alpha|,|\beta| \in F} \in M_{\vartheta_F}(\mathcal{O}_X^{(0)}).$$

Here m_k denotes the natural projection onto the subspace $\mathcal{O}_X^{(k)}$ of elements which transform like z^k under the gauge action. Let us apply the above construction to the parameters $11\delta^{\frac{1}{2}}$, n_0, and any $\Delta \in Pf(\mathcal{O}_X^{(0)})$, and find the corresponding F. Pick $(\tilde{\phi}, \psi, M_N) \in \operatorname{CPA}(\mathcal{D}, \Omega^{(n+n_0+\max F, X)}, \frac{\delta}{4\max_{p \in F} \vartheta_p{}^2})$ and extend $\tilde{\phi}$ by Arveson's theorem [3] to a u.c.p. map ϕ on $\mathcal{O}_X^{(0)}$, so that

$$(\phi, \psi, M_N) \in \operatorname{CPA}\Big(\mathcal{O}_X^{(0)}, \Omega^{(n+n_0+\max F, X)}, \frac{\delta}{4\max_{p \in F} \vartheta_p{}^2}\Big).$$

Since $I \in \Omega$, we have

$$\|(\iota - \psi\phi)(\rho_\mu(q_\beta))\| < \frac{\delta}{4\max_{p \in F} \vartheta_p{}^2}, \quad |\mu| + |\beta| \le n + n_0 + \max F - 1.$$

Also,

$$\|(\iota - \psi\phi)(\rho_\mu(t))\| < \frac{\delta}{4\max_{p \in F} \vartheta_p{}^2}, \quad |\mu| \le n + n_0 + \max F - 1.$$

By Lemma 6.8,

$$\|(\iota - \psi\phi)(\rho_\mu(q_\beta t q_\alpha))\| < \frac{11}{2\max_{p \in F} \vartheta_p} \delta^{\frac{1}{2}}$$

for $|\mu| + |\beta|, |\mu| + |\alpha| \le n + n_0 + \max F - 1$ and $t \in \Omega$. We set

$$\Omega'_n = \{\rho_\mu(q_\alpha t q_\beta), |\mu| + |\alpha|, |\mu| + |\beta| \le n - 1, t \in \Omega\},$$

and so $(\phi, \psi, M_N) \in \operatorname{CPA}(\Omega'^{(\max F, X)}_{n+n_0}, \frac{11}{2\max_{p \in F} \vartheta_p} \delta^{\frac{1}{2}})$. By the construction at the beginning we can find

$$(\phi', \psi', \mathcal{B}') \in \operatorname{CPA}_0(\cup_{|\gamma| \le n_0} \Omega'_{n+n_0} x_\gamma, 11\delta^{\frac{1}{2}}).$$

Consider the u.c.p. map $\psi_m : (t_{\mu\nu}) \to M_{\vartheta_m}(\mathcal{B}(\mathcal{H})) \to \sum_{|\mu|,|\nu|=m} x_\mu t_{\mu\nu} x_\nu^* \in \mathcal{B}(\mathcal{H})$ and the contractive c.p. map ϕ_m defined in Lemma 6.7. We claim that

$$(\phi'', \psi'', M_{\vartheta_{n+n_0-1}} \otimes \mathcal{B}') \in \mathrm{CPA}_0(\mathcal{O}_X, \Omega(n_0)^{(n)}, 11\vartheta_{n_0}\delta^{\frac{1}{2}}),$$

where $\psi'' := \psi_{n+n_0-1} \circ (\iota_{M_{\vartheta_{n+n_0-1}}} \otimes \psi')$ and $\phi'' := (\iota_{M_{\vartheta_{n+n_0-1}}} \otimes \phi') \circ \phi_{n+n_0-1}$. To establish the claim, note first that, for $j = 0, \dots, n-1$, $t \in \Omega$, and $|\beta| \le |\alpha| \le n_0$,

$$\|(\psi''\phi'' - \iota)(\theta^j(x_\alpha t x_\beta^*))\| \le \|\iota_{M_{\vartheta_{n+n_0-1}}} \otimes (\psi'\phi' - \iota) \circ \phi_{n+n_0-1} \circ \theta^j(x_\alpha t x_\beta^*)\|$$

Using notation from Lemma 6.7, the term on the right is bounded by

$$\max_{|\delta|=j} \sum_{|\nu'|=|\alpha|-|\beta|} \max_{|\mu'|} \|(\psi'\phi' - \iota)(\rho_{\mu'}(q_{\delta\alpha} t q_{\delta\beta}) x_\nu')\| \le 11\vartheta_{n_0}\delta^{\frac{1}{2}}.$$

Notice that the range of ϕ'' is a matrix algebra of rank $\vartheta_{n+n_0-1}\vartheta_F N$. However, we can reduce this rank by taking into account degeneracies, and so we will consider ϕ'' as a maps with range in matrices of rank $\sum_{p\in F} \vartheta_{n+n_0-1+p} N$. Assume for the moment that $a \ge 0$. Given $\epsilon > 0$ let $q \in \mathbb{N}$ be such that $\|q_\gamma a q_\gamma - a^{\frac{1}{2}} q_\gamma a^{\frac{1}{2}}\| < \epsilon$, $|\gamma| \ge q$. Then for all j,

$$q_\gamma \theta^j(a) q_\gamma = \sum_{|\beta|=j} x_\beta \rho_\beta(q_\gamma) a \rho_\beta(q_\gamma) x_\beta^*$$

$$\le \sum_{|\beta|=j} x_\beta a^{\frac{1}{2}} \rho_\beta(q_\gamma) a^{\frac{1}{2}} x_\beta^*$$

$$+ \left\| \sum_{|\beta|=j} x_\beta (\rho_\beta(q_\gamma) a \rho_\beta(q_\gamma) - a^{\frac{1}{2}} \rho_\beta(q_\gamma) a^{\frac{1}{2}}) x_\beta^* \right\|$$

$$\le \theta^j(a) + \epsilon.$$

We compute, for $n \ge \max F + n_0 + q$,

$$\mathrm{Tr}\ \exp(\phi''(a^{(n)}))$$

$$= \sum_{p\in F} \sum_{\alpha\in\Lambda^{(n+n_0-1+p)}} \mathrm{Tr}\ \exp(\tilde{\phi}(x_\alpha^* a^{(n)} x_\alpha))$$

$$\le \sum_{p\in F} \sum_{\gamma\in\Lambda^{(p+n_0+q)}} \sum_{\alpha\in\Lambda^{(n-q-1)}} \mathrm{Tr}\ \exp(\tilde{\phi}(x_\alpha^* x_\gamma^* a^{(n)} x_\gamma x_\alpha))$$

$$\le \sum_{p\in F} \left[\sum_{\gamma\in\Lambda^{(p+n_0+q)}} \exp((p+n_0+q)\|a\|) \right.$$

$$\left. \times \sum_{\alpha\in\Lambda^{(n-q-1)}} \mathrm{Tr}\ \exp(\tilde{\phi}(x_\alpha^* q_\gamma a^{(n-p-n_0-q)} q_\gamma x_\alpha)) \right], \tag{6.5}$$

where we have used the fact that, by the Peierls–Bogoliubov inequality (cf. Prop. 31.5 in [19]), for $p \in F$ and $|\gamma| = p + n_0 + q$,

$$\sum_{\alpha\in\Lambda^{(n-q-1)}} \mathrm{Tr}\ \exp(\tilde{\phi}(x_\alpha^* x_\gamma^* a^{(n)} x_\gamma x_\alpha))$$

$$\leq \exp{(n_0 + p + q)}\|a\| \sum_{\alpha \in \Lambda^{(n-q-1)}} \mathrm{Tr}\ \exp(\tilde{\phi}(x_\alpha^* q_\gamma a^{(n-p-n_0-q)} q_\gamma x_\alpha)).$$

Now (6.5) is bounded by

$$\sum_{p \in F} \vartheta_{p+n_0+q} \exp((p + n_0 + q)\|a\| + (n - p - n_0 - q)\epsilon)$$

$$\times \sum_{\alpha \in \Lambda^{(n-q-1)}} \mathrm{Tr}\ \exp(\tilde{\phi}(x_\alpha^* a^{(n-p-n_0-q)} x_\alpha))$$

$$\leq \sum_{p \in F} \vartheta_{p+n_0+q} \exp(2(p + n_0 + q)\|a\| + (n - p - n_0 - q)\epsilon)$$

$$\times \sum_{\alpha \in \Lambda^{(n-q-1)}} \mathrm{Tr}\ \exp(\tilde{\phi}(x_\alpha^* a^{(n-q)} x_\alpha)).$$

This shows that

$$P_\theta(\mathcal{O}_X, a, \Omega(n_0), 11\vartheta_{n_0}\delta^{\frac{1}{2}}) \leq P_X(\mathcal{D}, a) + \epsilon,$$

and so, by the arbitrariness of ϵ, $P_\theta(\mathcal{O}_X, a) \leq P_X(\mathcal{D}, a)$. For general a we write $a = a_+ - \lambda I$ with a_+ positive and $\lambda \in \mathbb{R}^+$. By Lemma 6.6 we have

$$P_\theta(a) = P_\theta(a_+) - \lambda \leq P_X(\mathcal{D}, a_+) - \lambda \leq P_X(\mathcal{D}, a),$$

and the proof is complete.   □

We next give a lower bound for $P_\theta(a)$.

Proposition 6.10. *Let a be a positive element of $\mathcal{D}$ such that there is $p \in \mathbb{N}$ for which*

$$[a,\ x_\alpha^* a x_\alpha] = 0, \tag{6.6}$$

and

$$[a,\ q_\alpha] = 0, \tag{6.7}$$

for $|\alpha| \geq p$. If $\sum_i^d q_i$ is invertible then

$$P_\theta(a) \geq \lim_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(n-1)}} e^{\|x_\alpha^* a x_\alpha\|}.$$

Proof. Suppose $b \in \mathcal{D}$ satisfies (6.6) and (6.7) for $|\alpha| \geq r$. Consider the C^*–subalgebra $\mathcal{C}(b, r)$ generated by

$$\{x_\alpha C^*(b, I) x_\alpha^*, \quad |\alpha| = nr, n = 0, 1, 2, \dots\}.$$

Notice that $\mathcal{C}(b, r)$ is θ^r–invariant. For fixed α with $|\alpha| = nr$, $x_\alpha C^*(b, I) x_\alpha^*$ is a commutative C^*–algebra. Furthermore, if $|\alpha| = hr$ and $|\beta| = kr$ with $h < k$, and $s, t \in \mathcal{C}^*(b, I)$, then $x_\alpha s x_\alpha^* x_\beta t x_\beta^*$ is nonzero only if $\beta = \alpha\beta'$ for some β' of length $(k - h)r \geq r$, and in this case

$$x_\alpha s x_\alpha^* x_\beta t x_\beta^* = x_\alpha s q_\alpha x_{\beta'} t x_\beta^* = x_\beta x_{\beta'}^* s x_{\beta'} q_\beta t x_\beta^*$$

$$= x_\beta t q_\beta x_{\beta'}^* s x_{\beta'} x_\beta^* = x_\beta t x_\beta^* x_\alpha s x_\alpha^*,$$

so that $\mathcal{C}(b,r)$ is commutative. Let T_r denote the epimorphism of the spectrum of $\mathcal{C}(b,r)$ obtained transposing θ^r. Consider the open (and closed) cover $\mathcal{U}$ of the spectrum of $\mathcal{C}(b,r)$ defined by the characteristic functions $\{x_\alpha x_\alpha^*, |\alpha| = r\}$. Then by the monotonicity of pressure (Prop. 3.3) and the fact that the noncommutative pressure reduces to the classical pressure on commutative C^*–algebras, we obtain

$$P_{\theta^r}(b) \geq p_{T_r}(b) \geq \lim_n \frac{1}{n} \sum_{\alpha \in \Lambda^{(rn-r)}} e^{\|x_\alpha^*(b+\theta^r(b)+\cdots+\theta^{r(n-1)}(b))x_\alpha\|}.$$

Suppose $a \in \mathcal{D}$ satisfies (6.6), (6.7) for $|\alpha| \geq p$ and set $a_r = a + \theta(a) + \cdots + \theta^{r-p}(a)$ for $r \geq p$. Then a_r satisfies the corresponding relations for $|\alpha| \geq r$. Therefore, for $r \geq p$,

$$P_{\theta^r}(a + \cdots + \theta^{r-p}(a)) \geq \lim_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(rn-r)}} e^{\|x_\alpha^* a_r^{(n)} x_\alpha\|}$$

where

$$a_r^{(n)} = (a + \cdots + \theta^{r-p}(a)) + (\theta^r(a) + \cdots + \theta^{2r-p}(a))$$
$$+ \cdots + (\theta^{r(n-1)}(a) + \cdots + \theta^{rn-p}(a)).$$

Now

$$\|x_\alpha^* a_r^{(n)} x_\alpha\| \geq \|x_\alpha^*(a + \cdots + \theta^{rn-1}(a))x_\alpha\| - n(p-1)\|a\|$$

and, by the monotonicity of pressure with respect to the self-adjoint element and scalar additivity (Prop. 3.1),

$$P_{\theta^r}(a + \cdots + \theta^{r-p}(a)) = P_{\theta^r}(a^{(r)} - (\theta^{r-p+1}(a) + \cdots + \theta^{r-1}(a)))$$
$$\leq r P_\theta(a) + (p-1)\|a\|$$

and so

$$P_\theta(a) + \frac{p-1}{r}\|a\| \geq \lim_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(n-r)}} e^{\|x_\alpha^* a^{(n)} x_\alpha\|} - \frac{p-1}{r}\|a\|.$$

Since

$$\lim_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(n-r)}} e^{\|x_\alpha^* a^{(n)} x_\alpha\|} = \lim_n \frac{1}{n} \log \sum_{\alpha \in \Lambda^{(n-1)}} e^{\|x_\alpha^* a^{(n)} x_\alpha\|},$$

for all r, we obtain the result letting $r \to \infty$. $\square$

Proof of Theorem 6.2. Combining Prop. 6.9 and Prop. 6.10, we obtain the proof of the first part of the Theorem. Assume that $a \in \mathcal{C}(\Lambda)$. Then, if $a \geq 0$, $\|x_\alpha^* a^{(n)} x_\alpha\|$ is the supremum of $a^{(n)}$ in the cylinder set $\{(i_j) \in \Lambda : (i_1, \ldots, i_{|\alpha|}) = \alpha\}$, thus the pressure formula reduces to the classical pressure formula for a positive continuous function on Λ (see, e.g., [11]). For the last part we will adapt from [18] a proof of the variational principle for a class of asymptotically Abelian C^*–algebras. By the additivity of pressure under addition of scalars, we may assume $a \geq 0$. Consider the unital,

commutative θ^r–invariant C^*–algebra $\mathcal{C}(a_r, r)$ introduced in the proof of Prop. 6.10. By the classical variational principle, given $\epsilon > 0$ there exists a θ^r–invariant state μ_r on $\mathcal{C}(a_r, r)$ such that

$$h_{\mu_r}(\theta^r \upharpoonright_{\mathcal{C}(a_r,r)}) + \mu_r(a_r) > P_{\theta^r \upharpoonright_{\mathcal{C}(a_r,r)}} - \epsilon.$$

By Prop. 4.17, μ_r extends to a θ^r–invariant state $\tilde{\sigma}_r$ on $\mathcal{O}_X$ in such a way that

$$h_{\tilde{\sigma}_r}(\theta^r) > h_{\mu_r}(\theta^r \upharpoonright_{\mathcal{C}(a_r,r)}) - 1.$$

Then $\sigma_r := \frac{1}{r} \sum_0^{r-1} \tilde{\sigma}_r \theta^j$ is θ–invariant. By Prop. 3.3 in [26] $h_{\sigma_r}(\theta) = \frac{1}{r} h_{\sigma_r}(\theta^r)$. By concavity of the Sauvageot–Thouvenot entropy (Prop. 4.18) and Lemma 4.19, one has

$$
\begin{aligned}
h_{\sigma_r}(\theta) = \frac{1}{r} h_{\sigma_r}(\theta^r) &\geq \frac{1}{r^2} \sum_0^{r-1} h_{\tilde{\sigma}_r \theta^j}(\theta^r) - \frac{\log r}{r} \\
&\geq \frac{1}{r} h_{\tilde{\sigma}_r}(\theta^r) - \frac{\log r}{r} \\
&> \frac{1}{r} h_{\mu_r}(\theta^r \upharpoonright_{\mathcal{C}(a_r,r)}) - \frac{1}{r} - \frac{\log r}{r}.
\end{aligned}
$$

Since

$$\sigma_r(a) \geq \frac{1}{r} \mu_r(a_r) - \frac{p-1}{r} \|a\|,$$

we infer that

$$\limsup_r h_{\sigma_r}(\theta) + \sigma_r(a) \geq \limsup_r \frac{1}{r} (P_{\theta^r \upharpoonright_{\mathcal{C}(a_r,r)}}(a_r) - \epsilon) = P_\theta(a).$$

The last equality has been proven in Prop. 6.10, taking into account Prop. 6.9.   □

2. Equilibrium states

We start this subsection proving that, at least if we restrict further the space of potentials, equilibrium states exist for $\mathcal{O}_X$. This generalizes Theorem 5.4 to the algebras $\mathcal{O}_X$.

Proposition 6.11. *Assume that $ht(\mathcal{A}, X) = 0$ and that a is a self-adjoint element of the commutative C^*–subalgebra $\mathcal{C}(\Lambda) \subset \mathcal{O}_X$, so, by Theorem 6.2, $P_\theta(a) = p_T(a)$. Then any faithful equilibrium measure for (Λ, T, a) extends to an equilibrium state for $(\mathcal{O}_X, \theta, a)$ for the CNT entropy, and thus also for the Sauvageot–Thouvenot entropy and the local state approximation entropy.*

Proof. By Prop. 8.4 and Lemma 8.3 in [21] any faithful shift-invariant measure μ on Λ extends to a σ–invariant state of $\mathcal{O}_X$ such that $h^{\mathrm{CNT}}{}_\sigma \geq h_\mu(T)$. Therefore if we start with an equilibrium measure for the Kolmogorov–Sinai entropy, by Prop. 4.10 and the fact that the Sauvageot–Thouvenot entropy majorizes the CNT entropy we obtain

$$
\begin{aligned}
hm_\sigma(\theta) + \sigma(a) \geq h_\sigma^{\mathrm{ST}}(\theta) + \sigma(a) &\geq \\
h_\sigma^{\mathrm{CNT}}(\theta) + \sigma(a) &\geq h_\mu(T) + \mu(a) = p_T(a) = P_\theta(a).
\end{aligned}
$$

□

We next give an upper bound for the local state approximation entropy of θ which is similar to the corresponding bound for the pressure (Prop. 6.9). This bound, together with Prop. 4.6, will lead, in a similar way, to a computation of $hm_\sigma(\theta)$, and will also be useful when discussing equilibrium states.

Proposition 6.12. *Let X be a Hilbert bimodule over a unital exact C^*–algebra $\mathcal{A}$ satisfying the same assumptions as in Prop. 6.9, let σ be a θ–invariant state of $\mathcal{O}_X$ and m the probability measure on Λ obtained restricting σ to $\mathcal{C}(\Lambda)$. Then*

$$hm_\sigma(\theta) \leq h_m(T) + ht(\mathcal{A}, X)$$

where $h_m(T)$ denotes the Kolmogorov–Sinai entropy of the shift T of Λ.

Proof. The proof parallels that of Prop. 6.9, with the same local approximations being employed to obtain an upper bound for $hm_\sigma(\theta)$. Thus, as in Prop. 6.9, given a finite subset $\Delta \subset \mathcal{O}_X^{(0)}$, $\delta > 0$, and $n_0 \in \mathbb{N}$, let $F = F(\delta, n_0) \subset \mathbb{N}$ be a finite subset independent of Δ such that for any $(\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\mathcal{O}_X^{(0)}, \Delta^{(\max F, X)}, \frac{11\delta^{\frac{1}{2}}}{2\max_F \theta})$ there is a triple

$$(\phi', \psi', \mathcal{B}') \in \mathrm{CPA}_0(\mathcal{O}_X, \cup_{|\gamma| \leq n_0} \Delta x_\gamma, 11\delta^{\frac{1}{2}})$$

where $\mathcal{B}' = M_{\sum_F \vartheta_p} \otimes \mathcal{B}$. Here we shall need recall, from Lemma 7.5 in [21], that ψ' is of the form $\psi' = \tilde{S}_{F,f} \circ (\iota \otimes \psi) : \mathcal{B}' \to \mathcal{B}(\mathcal{H})$ where $f \in \ell^2(\mathbb{N})$ has support in F, $\|f\|_2 \leq 1$, and

$$\tilde{S}_{F,f} : t = (t_{\alpha,\beta}) \in M_{\sum_F \vartheta_p}(\mathcal{B}(\mathcal{H})) \to \sum_{|\alpha|, |\beta| \in F} f(|\alpha|)\overline{f(|\beta|)}x_\alpha t_{\alpha,\beta}x_\beta^* \in \mathcal{B}(\mathcal{H}).$$

Let $\Omega \subset \mathcal{A}$ be a finite subset containing I. Pick

$$(\tilde{\phi}, \psi, M_N(\mathbb{C})) \in \mathrm{CPA}\left(\mathcal{A}, \Omega^{(n+n_0+\max F, X)}, \frac{\delta}{4\max^2_{p \in F} \vartheta_p}\right)$$

of minimal rank and follow the same procedure as in the proof of Prop. 6.9 to obtain a triple

$$(\phi'', \psi'', M_{\vartheta_{n+n_0-1}} \otimes M_{\sum_{p \in F} \vartheta_p} \otimes M_N) \in \mathrm{CPA}_0(\mathcal{O}_X, \Omega(n_0)^{(n)}, 11\vartheta_{n_0}\delta^{\frac{1}{2}})$$

where $\psi'' = \psi_{n+n_0-1} \circ (\iota_{M_{n+n_0-1}} \otimes \psi')$. Pick $\omega \in \mathfrak{E}(\sigma, \iota)$. The positive linear functional $\omega \circ \psi''$ is determined by it values on each matrix unit, which are given by

$$\omega \circ \psi''(e_{\alpha,\beta} \otimes e_{\mu,\nu} \otimes e_{i,j}) = f(|\mu|)\overline{f(|\nu|)}\omega(x_{\alpha\mu}\psi(e_{i,j})x_{\beta\nu}^*).$$

We have that

$$S(\omega \circ \psi'') \leq S(\mathrm{diag}(|f(|\mu|)|^2\omega(x_{\alpha\mu}\psi(e_{i,i})x_{\alpha\mu}^*)_{|\alpha|=n+n_0-1, |\mu|\in F, i=1,\ldots,N}))$$

by the estimate on page 60 in [19], e.g., and this last expression is bounded above by

$$S\left(\mathrm{diag}\left(\frac{1}{N}|f(|\mu|)|^2 m(p_{\alpha\mu})\right)_{\alpha,\mu,i}\right),$$

which in turn is bounded above by

$$-\sum_{\alpha,\mu}|f(|\mu|)|^2 m(p_{\alpha\mu})\log\left(\frac{1}{N}|f(|\mu|)|^2 m(p_{\alpha\mu})\right)$$

$$=\sum_{\alpha,\mu}-|f(|\mu|)|^2 m(p_{\alpha\mu})\log(m(p_{\alpha\mu}))+\log N\sum_{\mu}|f(|\mu|)|^2 m(p_\mu)$$

$$-\sum_{\mu}|f(|\mu|)|^2 m(p_\mu)\log m(p_\mu),$$

with the equality following from the T–invariance of m. Finally, using the equality

$$\sum_{\mu}|f(|\mu|)|^2 m(p_{\alpha\mu})=\|f\|_2^{\;2}m(p_\alpha)$$

and the concavity of $x\mapsto -x\log x$, we see that the last displayed expression is bounded by

$$-\sum_{|\alpha|=n+n_0-1}\|f\|_2^{\;2}m(p_\alpha)\log(\|f\|_2^{\;2}m(p_\alpha))$$

$$+\log N\sum_{|\mu|\in F}|f(|\mu|)|^2 m(p_\mu)-\sum_{|\mu|\in F}|f(|\mu|)|^2 m(p_\mu)\log m(p_\mu).$$

Therefore, since $\|f\|_2\le 1$, Prop. 4.5 yields

$$hm_\sigma(\theta,\iota,\omega,\Omega(n_0))\le h_\mu(T)+ht(\mathcal{A},X).$$

$\square$

We next derive a few consequences on equilibrium states from the previous proposition. There is a natural conditional expectation $E:\mathcal{O}_X\to\mathcal{D}_0$ where $\mathcal{D}_0$ is the C^*–subalgebra of $\mathcal{D}$ generated by elements of the form $x_\alpha a x_\alpha^*$, $a\in\mathcal{A}$, $\alpha\in\cup_n\Lambda^{(n)}$, defined in the following way. Compose the average over the gauge action $\mathcal{O}_X\to\mathcal{O}_X^{(0)}$ with the pointwise norm limit $P:\mathcal{O}_X\to\mathcal{O}_X^{(0)}$ of the maps

$$t\to P_n(t)=\sum_{|\alpha|=n}x_\alpha x_\alpha^* t x_\alpha x_\alpha^*.$$

One has

$$E\circ\theta=\theta\circ E.$$

Corollary 6.13. *Let $(\mathcal{O}_X,\theta)$ be the C^*–dynamical system constructed as in Theorem 6.2. Assume that $ht(\mathcal{A},X)=0$. If σ is a θ–invariant state and the restriction m of σ to $\mathcal{C}(\Lambda)$ is a faithful measure, then $\sigma\circ E$ is a θ–invariant state centralized by $\mathcal{C}(\Lambda)$ for which*

$$h_m(T)=hm_{\sigma\circ E}(\theta)=h_{\sigma\circ E}^{ST}(\theta)=h_{\sigma\circ E}^{CNT}(\theta),$$

where h^{ST} and h^{CNT} denote respectively the Sauvageot–Thouvenot and CNT entropy. If moreover $a\in\mathcal{D}_0$ is a self–adjoint element and σ is an equilibrium state for $(\mathcal{O}_X,\theta,a)$, then $\sigma\circ E$ is an equilibrium state for the same system (both with respect to hm).

Proof. Note that $\sigma \circ E$ is a θ–invariant state since E commutes with θ. Furthermore $\sigma \circ E$ is centralized by $\mathcal{C}(\Lambda)$ since E is a conditional expectation onto $\mathcal{D}_0$, which contains $\mathcal{C}(\Lambda)$, and $\mathcal{C}(\Lambda)$ commutes with $\mathcal{D}_0$. Therefore Props. 8.2 and 8.3 in [21] can be applied to $\sigma \circ E$. Using, Prop. 6.12, Prop. 4.10, the fact that Sauvageot–Thouvenot entropy majorizes the CNT entropy [26], and Props. 8.2 and 8.3 of [21], respectively, we infer that

$$h_m(T) \geq hm_{\sigma \circ E}(\theta) \geq h^{\mathrm{ST}}_{\sigma \circ E}(\theta) \geq h^{\mathrm{CNT}}_{\sigma \circ E}(\theta) \geq h_m(T).$$

Assume now that σ is an equilibrium state for $(\mathcal{O}_X, \theta, a)$. Then

$$hm_{\sigma \circ E}(\theta) + \sigma(a) = h_m(T) + \sigma(a) \geq hm_\sigma(\theta) + \sigma(a) = P_\theta(a),$$

and so $\sigma \circ E$ must be an equilibrium state for the same system as well. $\square$

The following is a converse of Prop. 6.11.

Corollary 6.14. *Let $(\mathcal{O}_X, \theta)$ be the C^*–dynamical system constructed as in Theorem 6.2. Assume that $ht(\mathcal{A}, X) = 0$ and let a be a selfadjoint element of the canonical Abelian subalgebra $\mathcal{C}(\Lambda)$. Let $H_\sigma(\theta)$ be either the local state approximation entropy, the Sauvageot–Thouvenot entropy, or the CNT entropy. If σ is a θ–invariant equilibrium state for $(\mathcal{O}_X, \theta, a)$ w.r.t. $H_\sigma(\theta)$, then the measure m obtained restricting σ to $\mathcal{C}(\Lambda)$ is an equilibrium measure for (Λ, T, a). Furthermore one has*

$$H_\sigma(\theta) = h_m(T)$$

where h_m is the Kolmogorov–Sinai entropy.

Proof. By the comparison between the various state–based entropies (Prop. 4.10), the fact that $P_\theta(a)$ coincides with the classical pressure of a (Prop. 6.2), Prop. 3.6, and Prop. 6.12 under the assumption $ht(\mathcal{A}, X) = 0$, we have

$$p_T(a) = P_\theta(a) = H_\sigma(\theta) \leq hm_\sigma(\theta) + \sigma(a) \leq h_m(T) + m(a),$$

so that m is an equilibrium state for (Λ, T, a). Since all the inequalities become equalities, we conclude that $H_\sigma(\theta) = h_m(T)$. $\square$

3. An application to Matsumoto algebras associated to subshifts

We conclude this section with an application to Cuntz–Krieger algebras, or, more generally, to Matsumoto C^*–algebras.

Corollary 6.15. *Let $\mathcal{O}_\Lambda$ denote the Matsumoto algebra associated to a subshift of one of the following kinds:*

a) *finite type subshifts,*

b) *sofic subshifts,*

c) *β–shifts.*

Then for any real-valued $f \in \mathcal{C}(\Lambda) \subset \mathcal{O}_\Lambda$, $P_\theta(f)$ equals the classical pressure of f with respect to the shift T:

$$P_\theta(f) = p_T(f).$$

Furthermore any shift-invariant measure μ on $\mathcal{C}(\Lambda)$ extends to a θ–invariant state σ on $\mathcal{O}_\Lambda$ with the propery $h_\sigma(\theta) \geq h_\mu(T)$. In particular, if μ is an equilibrium measure for (Λ, T, f), the corresponding extension is an equilibrium state for $(\mathcal{O}_\Lambda, \theta, f)$.

Proof. For Matsumoto C^*–algebras the coefficient algebra $\mathcal{A}$ is commutative and commutes with $\mathcal{C}(\Lambda)$. Furthermore, the growth of the local completely positive δ–ranks $rcp(\mathcal{A}, \Omega^{(n,X)}, \delta)$ is polynomial (see[21]), and so $ht(\mathcal{A}, X) = 0$, implying the first part of the assertion. Let μ be a T–invariant measure on Λ. That μ extends to a θ–invariant state σ on $\mathcal{O}_\Lambda$ with entropy as least as large has been proven in Theorem 8.6 of [21]. The rest is now clear. $\square$

7   The KMS condition and equilibrium in $\mathcal{O}_A$

We will show how certain equilibrium states for Cuntz–Krieger algebras can be constructed from KMS states with respect to a suitable one-parameter automorphism group in the case where the self-adjoint element has small variation on the underlying subshift of finite type and is a Hölder continuous function. The key idea is to establish a connection between KMS states with respect to this group and the Perron–Frobenius–Ruelle theorem for subshifts of finite type [23, 5, 30].

Let A be $\{0, 1\}$-matrix with no row or column identically zero, and let $a \in \mathcal{C}(\Lambda_A)$ be a self-adjoint element. Consider for $\beta \in \mathbb{R}$ the unitary group of $\mathcal{C}(\Lambda_A)$, $U_{\beta,a}(t) = \exp(it(\beta - a))$, and define the one-parameter automorphism group of $\mathcal{O}_A$

$$\alpha^{\beta,a}{}_t(s_i) = U_{\beta,a}(t)s_i, \quad i = 1, \ldots, d,$$

where $s_1, \ldots, s_d$ are the generating partial isometries of $\mathcal{O}_A$ [33]. We shall also need a positive operator of $\mathcal{C}(\Lambda_A)$ whose spectral properties and their relation with equilibrium states were first studied by Ruelle [23] in the case of the full 2–shift and in a more general setting by Bowen [5] and Walters [30]. Set

$$\mathcal{L}_a(f)(x) = \sum_{i:A_{ix_1}=1} \exp(a(ix))f(ix), \quad f \in \mathcal{C}(\Lambda_A),$$

where $x = (x_k)_k \in \Lambda_A$. Notice that we can write, in $\mathcal{O}_A$,

$$\mathcal{L}_a(f) = \sum_i s_i^* e^a f s_i.$$

Thus $\mathcal{L}_a$ extends in an obvious way to an operator on $\mathcal{O}_A$, which we will denote by $\overline{\mathcal{L}}_a$. We begin by establishing some partial results.

Lemma 7.1. *$\mathcal{C}(\Lambda_A)$ is contained in the algebra of fixed points under $\alpha^{\beta,a}$ for all $\beta \in \mathbb{R}$ and all self-adjoint $a \in \mathcal{C}(\Lambda_A)$.*

Proof. For all $j \in \mathbb{N}$ and $t \in \mathbb{R}$ we have $\theta^j(U_{\beta,a}(t)) \in \mathcal{C}(\Lambda_A)$, and therefore $[\theta^j(U_{\beta,a}(t)), \mathcal{C}(\Lambda_A)] = 0$. On the other hand, if $f \in \mathcal{C}(\Lambda_A)$ is of the form $f = s_{i_1} \ldots s_{i_r}(s_{i_1} \ldots s_{i_r})^*$, then for all $t \in \mathbb{R}$

$$\alpha^{\beta,a}{}_t(f) = U_{\beta,a}(t) \ldots \theta^{r-1}(U_{\beta,a}(t)) f \theta^{r-1}(U_{\beta,a}(-t)) \ldots U_{\beta,a}(-t) = f$$

since θ is multiplicative on $\mathcal{C}(\Lambda_A)$ (in fact, one can easily check that θ is multiplicative on the relative commutant of $\{s_1^* s_1, \ldots, s_d^* s_d\}$ in $\mathcal{O}_A$), completing the the proof. $\qquad\square$

The following result is well-known. We supply a proof for convenience.

Lemma 7.2. *If μ is a state on $\mathcal{C}(\Lambda_A)$ such that, for some $\beta \in \mathbb{R}$,*

$$e^\beta \mu(f) = \mu(\mathcal{L}_a(f)), \quad f \in \mathcal{C}(\Lambda_A),$$

then

$$\min(a) + \log r(A) \le \beta \le \max(a) + \log r(A).$$

Proof. For $n \in \mathbb{N}$ and $x = (x_k)_k \in \Lambda_A$,

$$\mathcal{L}_a^n(1)(x) = \sum_{A_{i_1 i_2} = \cdots = A_{i_n x_1} = 1} e^{a^{(n)}(i_1, \ldots, i_n, x)}.$$

Therefore

$$e^{n \min a} \vartheta_n \le \mathcal{L}_a^n(1) \le e^{n \max a} \vartheta_n,$$

where, as usual, ϑ_n denotes the cardinality of the set of words of length n appearing in Λ_A. Applying μ yields

$$e^{n \min a} \vartheta_n \le e^{n\beta} \le e^{n \max a} \vartheta_n,$$

and so computing $\lim_n \frac{1}{n} \log(\cdot)$ and using the fact that $\lim_n \frac{1}{n} \log \vartheta_n = \log r(A)$ (see, e.g., [11]) we obtain the desired estimate. $\qquad\square$

We next describe a bijective correspondence between KMS states for $\alpha^{\beta,a}$ and positive eigenvectors of the Banach space adjoint of the Ruelle operator

$$\mathcal{L}_a^* : \mathcal{C}(\Lambda_A)^* \to \mathcal{C}(\Lambda_A)^*.$$

We start showing that $(\alpha^{\beta,a}, 1)$–KMS states restrict to positive eigenvectors of $\mathcal{L}_a^*$.

Lemma 7.3. *If ω is a $(\alpha^{\beta,a}, 1)$–KMS state on $\mathcal{O}_A$ then*

$$\sum_i \omega(s_i^* e^a b s_i) = e^\beta \omega(b), \quad b \in \mathcal{O}_A.$$

In particular, if $\mu := \omega \restriction \mathcal{C}(\Lambda_A)$ then

$$\mathcal{L}_a^*(\mu) = e^\beta \mu.$$

Proof. We first note that, for $i = 1, \ldots, d$, $\alpha^{\beta,a}{}_{-i}(s_j^*) = s_j^* e^{-\beta+a}$. Thus, by the KMS property, if $b \in \mathcal{O}_A$ then

$$\omega(b) = \sum_j \omega(bs_j s_j^*) = \sum_j \omega(\alpha^{\beta;a}_{-i}(s_j^*)bs_j) = e^{-\beta} \sum_j \omega(s_j^* e^a b s_j).$$

□

Lemma 7.4. *If A is aperiodic then $r(A) > 1$.*

Proof. Let N be a positive integer such that all of the entries of A^N are positive. Since $A \in M_d(\{0,1\})$, $A_{ij}^N \geq 1$ for all i,j, and therefore $A_{ij}^{pN} \geq d^{p-1}$ for $p \in \mathbb{N}$, so that

$$r(A) = \lim_n \|A^n\|^{1/n} = \lim_p \|A^{Np}\|^{1/Np} \geq d^{1/N} > 1.$$

□

We define a metric on Λ_A by $d(x,y) = \frac{1}{k}$ where k is the least integer for which $x_k \neq y_k$. For $f \in \mathcal{C}(\Lambda_A)$, we set

$$\mathrm{var}_0(f) = \max f - \min f,$$

$$\mathrm{var}_n(f) = \sup\left\{|f(x) - f(y)|, d(x,y) \leq \frac{1}{n+1}\right\}, \quad n \in \mathbb{N}.$$

Note that contunuity implies that $\mathrm{var}_n(f) \to 0$ as $n \to \infty$.

Lemma 7.5. *Let A be an aperiodic $\{0,1\}$–matrix and $a \in \mathcal{C}(\Lambda_A)$ a self-adjoint element such that $\mathrm{var}_0(a) < \log r(A)$. Then any $(\alpha^{\beta,a}, 1)$–KMS state on $\mathcal{O}_A$ is gauge-invariant and faithful.*

Proof. We first establish gauge invariance. Let ω be a $(\alpha^{\beta,a}, 1)$–KMS state. We need to show that if $b \in \mathcal{O}_A^{(k)}$ for some $k > 0$, then $\omega(b) = 0$. It suffices to pick b nonzero and of the form $b = s_{i_1} \ldots s_{i_s}(s_{j_1} \ldots s_{i_r})^*$ with $s - r = k$. Since ω is an $\alpha^{\beta,a}$–KMS state, it is $\alpha^{\beta,a}$–invariant, and therefore

$$\omega(\alpha^{\beta,a}{}_{it}(b)) = \omega(b), \quad t \in \mathbb{R},$$

that is,

$$e^{-k\beta t}\omega(e^{ta} \ldots \theta^{s-1}(e^{ta})b\theta^{r-1}(e^{-ta}) \ldots e^{-ta}) = \omega(b).$$

Now by Lemma 7.1 $\theta^j(e^{ta})$ lies in the centralizer of ω, and thus the above equality reduces to

$$e^{-k\beta t}\omega(\theta^r(e^{ta}) \ldots \theta^{s-1}(e^{ta})b) = \omega(b),$$

and so $|\omega(b)| \leq e^{-k\beta t}\|e^{ta}\|^k\|b\|$. Hence, for $t \geq 0$,

$$\frac{|\omega(b)|}{\|b\|} \leq e^{kt(-\beta+\max a)},$$

which implies, assuming $\omega(b) \neq 0$, that $\beta \leq \max a$. On the other hand, by Lemma 7.3 the restriction of ω to $\mathcal{C}(\Lambda_a)$ is a positive eigenvector of the transposed Ruelle operator with eigenvalue e^β, and so

$\beta \geq \min a + \log r(A)$ by Lemma 7.2. Therefore we must have $\log r(A) \leq \mathrm{var}_0(a)$, which contradicts our assumption.

We next show that ω is faithful. By the previous part, it suffices to show that the restriction of ω to $\mathcal{O}_A^{(0)}$ is faithful. Set $\mathfrak{I} = \{b \in \mathcal{O}_A^{(0)} : \omega(b^*b) = 0\}$. Clearly $\mathfrak{I}$ is a closed left ideal of $\mathcal{O}_A^{(0)}$. If c ranges over a dense set of analytic vectors for $\alpha^{\beta,a}$ and $b \in \mathfrak{I}$, then by the KMS property

$$\omega(c^*b^*bc) = \omega(\alpha_{-i}^{\beta,a}(c)c^*b^*b) = 0,$$

so that $\mathfrak{I}$ is a two-sided closed ideal of $\mathcal{O}_A^{(0)}$. Since A is aperiodic, $\mathcal{O}_A^{(0)}$ is a simple AF C^*–algebra, and so $\mathfrak{I} = 0$. $\qquad\square$

Proposition 7.6. *Let A be an aperiodic $\{0,1\}$–matrix. If $a \in \mathcal{C}(\Lambda_A)$ is a self-adjoint element the map*

$$\omega \rightarrow \mu := \omega \upharpoonright \mathcal{C}(\Lambda_A)$$

sets up a surjective correspondence between the set of $(\alpha^{\beta,a}, 1)$–KMS states of $\mathcal{O}_A$ and the set of probability measures on Λ_A for which

$$\mathcal{L}_a^* \mu = e^\beta \mu.$$

If in addition $\mathrm{var}_0(a) < \log r(A)$, this map is a bijection.

Proof. Let ω be a $(\alpha^{\beta,a}, 1)$–KMS state. By Lemma 7.3 the measure μ corresponding to $\omega \upharpoonright \mathcal{C}(\Lambda_A)$ is an eigenvector of the transposed Ruelle operator with eigenvalue e^β.

We show that any state μ on $\mathcal{C}(\Lambda_A)$ arising as an eigenvector of the transposed Ruelle operator with eigenvalue e^β is the restriction of a (gauge–invariant) $(\alpha^{\beta,a}, 1)$–KMS state. Consider the C^*–subalgebras F_n of $\mathcal{O}_A$ linearly spanned by elements of the form $s_{i_1} \ldots s_{i_n} \mathcal{C}(\Lambda_A)(s_{j_1} \ldots s_{j_n})^*$. Note that $F_n \subset F_{n-1}$ and that $\cup_n F_n$ is dense in $\mathcal{O}_A^{(0)}$. Recursively define for each $n = 0, 1, 2, \ldots$ a state ω_n on F_n by $\omega_0 = \mu$ and

$$\omega_n(b) = e^{-\beta} \sum_i \omega_{n-1}(s_i^* e^{a/2} b e^{a/2} s_i), \quad b \in F_n.$$

Then ω_1 extends ω_0, and one can check that ω_n extends ω_{n-1} for all n. Consider the gauge-invariant state ω of $\mathcal{O}_A$ that extends ω_n on F_n. By construction, ω satisfies the scaling property

$$\sum_i \omega(s_i^* e^a b s_i) = e^\beta \omega(b), \quad b \in \mathcal{O}_A.$$

We show that ω is a $(\alpha^{\beta,a}, 1)$–KMS state. Consider elements of the form $b = s_{i_1} \ldots s_{i_s}(s_{j_1} \ldots s_{j_r})^*$, $c = s_{h_1} \ldots s_{h_r}(s_{k_1} \ldots s_{k_s})^*$. We need to show that $\omega(bc) = \omega(\alpha_{-i}^{\beta,a}(c)b)$. If $(j_1, \ldots, j_r) \neq (h_1, \ldots, h_r)$ then $\omega(bc) = 0$. We also have $\omega(\alpha_{-i}^{\beta,a}(c)b) = 0$ since the l.h.s. is

$$\omega(e^{\beta-a} s_{h_1} \ldots e^{\beta-a} s_{h_r} s_{k_s}^* e^{-\beta+a} \ldots s_{k_1}^* e^{-\beta+a} s_{i_1} \ldots s_{i_s}(s_{j_1} \ldots s_{j_r})^*),$$

which, by an iteration of the scaling property, is seen to be zero. Assume now that $(j_1, \ldots, j_r) = (h_1, \ldots, h_r)$. The scaling property tells us again that if $\omega(bc)$ is nonzero, we must have $(i_1, \ldots, i_s) =$

$(k_1, \ldots, k_s)$. Clearly, the above computation shows that if $\omega(\alpha_{-i}^{\beta;a}(c)b) \neq 0$ the same condition holds. Again, by the scaling property, the latter is

$$\omega(s_{i_s}^* e^{-\beta+a} \cdots s_{i_1}^* e^{-\beta+a} s_{i_1} \cdots s_{i_s} (s_{j_1} \cdots s_{j_r})^* s_{j_1} \cdots s_{j_r})$$
$$= \omega(s_{i_1} \cdots s_{i_s} (s_{j_1} \cdots s_{j_r})^* s_{j_1} \cdots s_{j_r} (s_{i_1} \cdots s_{i_s})^*)$$
$$= \omega(bc).$$

We next show that the map $\omega \to \mu$ is one-to-one if $\mathrm{var}_0(a) < \log r(A)$. Since ω is gauge-invariant by Lemma 7.5, it is determined by its restriction to $\mathcal{O}_A^{(0)}$. Applying $\overline{\mathcal{L}}_a^n$ on words of the form $s_{i_1} \ldots s_{i_n} (s_{j_1} \ldots s_{j_n})^*$ yields an element of $\mathcal{C}(\Lambda_A)$, and so again by Lemma 7.3 the restriction of ω to $\mathcal{O}_A^{(0)}$ is determined uniquely by its values on $\mathcal{C}(\Lambda_A)$, and the proof is complete. $\square$

We recall the Perron–Frobenius–Ruelle theorem for subshifts of finite type.

Theorem 7.7. [23, 5, 30] *Let A be an aperiodic $\{0,1\}$–matrix and $a \in \mathcal{C}(\Lambda_A)$ a self-adjoint element satisfying*

$$\sum_n \mathrm{var}_n(a) < \infty.$$

Then

(a) *$\mathcal{L}_a$ admits a strictly positive eigenvector h which is unique up to a scalar factor,*

(b) *$\mathcal{L}_a^*$ admits a unique probability measure eigenvector μ,*

(c) *one has $\mathcal{L}_a h = \lambda h$ and $\mathcal{L}_a^* \mu = \lambda \mu$, where $\log \lambda = p_T(a) = \log r(\mathcal{L}_a)$, with $r(\mathcal{L}_a)$ the spectral radius of $\mathcal{L}_a$,*

(d)
$$\frac{\mathcal{L}_a^n(f)}{\lambda^n} \to \frac{\mu(f)}{\mu(h)} h$$

uniformly for all $f \in \mathcal{C}(\Lambda_A)$,

(e) *$\nu(f) := \mu(hf)$, $f \in \mathcal{C}(\Lambda_A)$ is the unique equilibrium measure for (Λ_A, T, a),*

(f) *μ an ν are faithful.*

We are now in a position to establish a connection between $(\alpha^{\beta,a}, 1)$–KMS states and equilibrium measures for $\mathcal{O}_A$.

Theorem 7.8. *Assume that A is aperiodic and that the self-adjoint element a satisfies*

$$\sum_n \mathrm{var}_n(a) < \infty.$$

Then $\mathcal{O}_A$ admits a $(\alpha^{\beta,a}, 1)$–KMS state if and only if $\beta = P_\theta(a)$. If $\mathrm{var}_0(a) < \log r(A)$ then there is exactly one such state ω. If h is the unique strictly positive eigenvector of $\mathcal{L}_a$ with $\omega(h) = 1$, $\sigma(b) := \omega(hb)$ is a faithful equilibrium state for $(\mathcal{O}_A, \theta, a)$.

Proof. By Prop. 7.6 the set of $(\alpha^{\beta,a}, 1)$–KMS states corresponds surjectively to the set of probability measures eigenvectors of $\mathcal{L}_a^*$ with eigenvalue e^β, and therefore by Theorem 7.7, there is a $(\alpha^{\beta,a}, 1)$–KMS state if and only if $\beta = P_\theta(a)$. If $\mathrm{var}_0(a) < \log r(A)$, there is exactly one such state, again by Prop. 7.6 and Theorem 7.7. Furthermore, by (e) of Theorem 7.7, the restriction of σ to $\mathcal{C}(\Lambda_A)$ is the unique equilibrium measure ν for (Λ_A, T, a). We note that σ is θ–invariant, for if $b \in \mathcal{O}_A$ then

$$\sigma(\theta(b)) = \sum_i \omega(hs_i b s_i^*) = e^{-P_\theta(a)} \sum_i \omega(s_i^* e^a h s_i b) = \omega(hb) = \sigma(b).$$

Since ω contains $\mathcal{C}(\Lambda_A)$ in its centralizer, the same holds for σ. Thus, by Lemma 5.3, $h_\mu(T) \leq h_\sigma(\theta)$, and hence

$$p_T(a) = h_\nu(T) + \nu(a) \leq h_\sigma(\theta) + \sigma(a) \leq P_\theta(a) = p_T(a),$$

which establishes that σ is an equilibrium state for $(\mathcal{O}_A, \theta, a)$. $\qquad\qquad\Box$

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